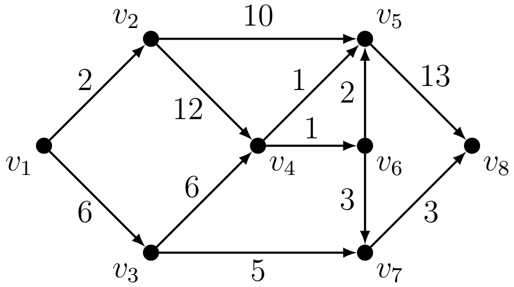
MTH6105 – Algorithmic Graph Theory — Spring 2024

Problem Sheet 9 — F. Fischer

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

---

1. Consider the following directed network.



(a) Apply the Ford-Fulkerson algorithm to the network, drawing the residual network after each iteration.

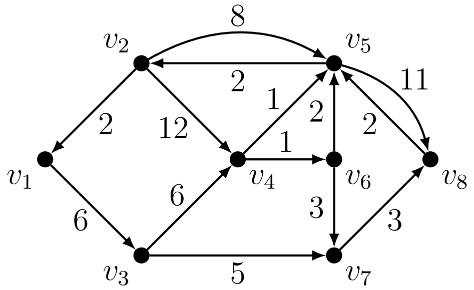
(b) Give a maximum v_1–v_8-flow of the network.

(c) Prove that the v_1–v_8-flow you have found is indeed a maximum v_1–v_8-flow.

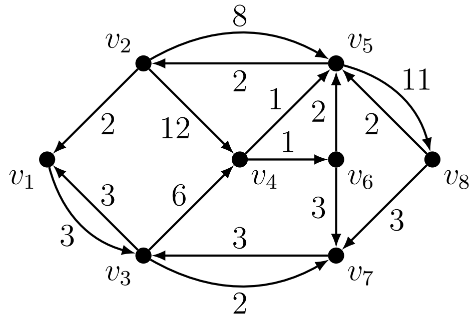
Solution:

(a) To find a maximum v_1–v_8-flow, we should run the Ford-Fulkerson algorithm for $s = v_1$ and $t = v_8$. We can start the algorithm from an arbitrary flow of the network and the corresponding residual network. If we start from the trivial flow that is zero for every arc, the residual network will be equal to the network itself. The algorithm then repeatedly finds a directed v_1–v_8-path P in the residual network and increases flow along this path by the minimum residual capacity $c_f(P)$ along the path. The algorithm terminates when the residual network does not contain a directed v_1–v_8-path. We may for example obtain the following augmenting paths and residual networks.

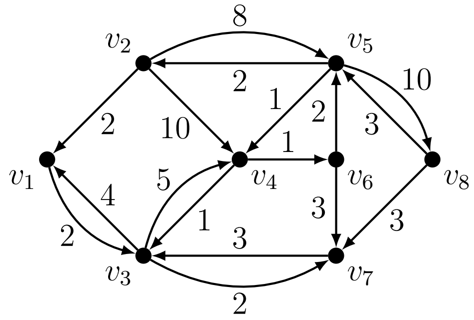
$P = v_1, v_2, v_5, v_8$ $\qquad$ $c_f(P) = 2$



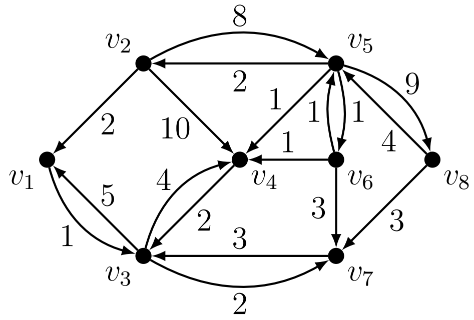
$P = v_1, v_3, v_7, v_8$ $\qquad c_f(P) = 3$



$P = v_1, v_3, v_4, v_5, v_8$ $\qquad c_f(P) = 1$



$P = v_1, v_3, v_4, v_6, v_5, v_8$ $\qquad c_f(P) = 1$



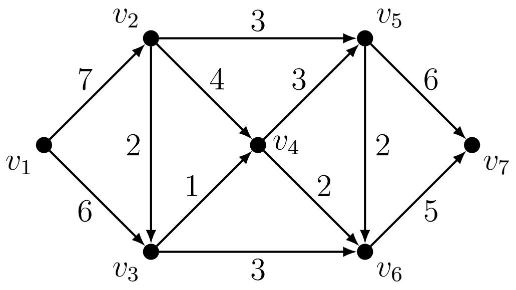
(b) In each iteration of the Ford-Fulkerson algorithm, the current flow f is augmented by sending an additional amount of $c_f(P)$ along an augmenting path P. The value of f at the point when the algorithm terminates can be obtained by adding for each arc of the original network the amounts of flow sent along it (note that some of these amounts may be negative, namely when an arc is a backward arc on an augmenting path). We thus obtain the flow f with

$$\begin{array}{llll} f(v_1v_2) = 2, & f(v_1v_3) = 5, & f(v_2v_4) = 0, & f(v_2v_5) = 2, \\ f(v_3v_4) = 2, & f(v_3v_7) = 3, & f(v_4v_5) = 1, & f(v_4v_6) = 1, \\ f(v_5v_8) = 4, & f(v_6v_5) = 1, & f(v_6v_7) = 0, & f(v_7v_8) = 3. \end{array}$$

(c) Consider the last residual network, and let $S = \{v_1, v_3, v_4, v_7\}$ be the set of vertices s such that there exists a directed v_1–s-path in this network. Note that $v_1 \in S$ and $v_8 \notin S$, so S is a v_1–v_8-cut of the original network. Moreover $C(S) = c(v_1v_2) + c(v_4v_5) + c(v_4v_6) + c(v_7v_8) = 7 = |f|$. By Corollary 6.5 in the lecture notes, f is a maximum v_1–v_8-flow and S a minimum v_1–v_8-cut.

---

2. Consider the following directed network.



Let g be the v_1—v_7-flow of this network with

$$
\begin{array}{llll}
g(v_1v_2) = 7, & g(v_1v_3) = 2, & g(v_2v_3) = 2, & g(v_2v_4) = 4, \\
g(v_2v_5) = 1, & g(v_3v_4) = 1, & g(v_3v_6) = 3, & g(v_4v_5) = 3, \\
g(v_4v_6) = 2, & g(v_5v_6) = 0, & g(v_5v_7) = 4, & g(v_6v_7) = 5.
\end{array}
$$

(a) Prove or disprove that g is a maximum v_1—v_7-flow of the network.

(b) Imagine that $c(v_3v_6)$ is decreased from 3 to 1. Does this affect the size of a maximum flow? Justify your answer.

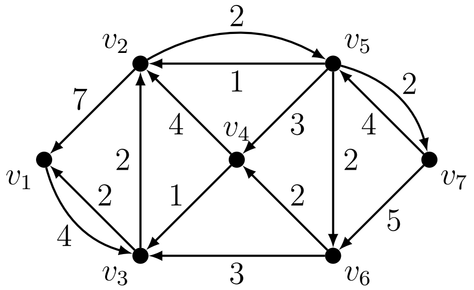
(c) Imagine that $c(v_3v_6)$ is increased from 3 to 4. Does this affect the size of a maximum flow? Justify your answer.

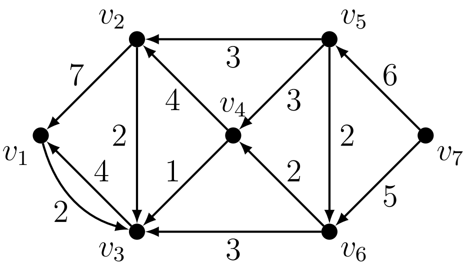
Solution:

(a) If we draw the residual network for g, we see that $Q = v_1, v_3, v_2, v_5, v_7$ is a g-augmenting v_1—v_7-path. We can thus augment g by sending an additional flow of $c_g(P) = 2$ along P and obtain a v_1—v_7-flow h with

$$
\begin{array}{llll}
h(v_1v_2) = 7, & h(v_1v_3) = 4, & h(v_2v_3) = 0, & h(v_2v_4) = 4, \\
h(v_2v_5) = 3, & h(v_3v_4) = 1, & h(v_3v_6) = 3, & h(v_4v_5) = 3, \\
h(v_4v_6) = 2, & h(v_5v_6) = 0, & h(v_5v_7) = 6, & h(v_6v_7) = 5.
\end{array}
$$

Since $|g| = 9$ and $|h| = 11$, g is not a maximum v_1—v_7-flow. The residual networks before and after augmentation look as follows.





(b) Let $S = \{v_1, v_3\}$ be the set of vertices s such that there exists a directed v_1—s-path in the second residual network. S is a v_1—v_7-cut of the original network, and $C(S) = c(v_1v_2) + c(v_3v_4) + c(v_3v_6) = 11 = |h|$. Thus, by Corollary 6.5 in the lecture notes, S is a minimum v_1—v_7-cut. Decreasing $c(v_3v_6)$ by 2 decreases $C(S)$ by 2, and the capacity of any other v_1—v_7-cut by at most 2. S will thus remain a minimum v_1—v_7-cut, which means that the size of a maximum v_1—v_7-flow also decreases by 2.

(c) Increasing $c(v_3v_6)$ by 1 increases $C(S)$ by 1, but it does not necessarily increase the size of a minimum v_1–v_7-cut. Indeed, the v_1–v_7-cut $T = \{v_1, v_2, v_3, v_4, v_6\}$ has capacity $C(T) = c(v_2v_5) + c(v_4v_5) + c(v_6v_7) = 11$, and this capacity does not depend on $c(v_3v_6)$.
