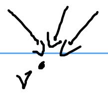
6. Network Flows

Consider a directed network (D, c). The weight $c(e)$ of arc e is now the capacity of e.

6.1 Maximum Flows

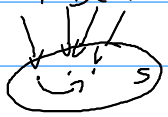
$A_D^-(v)$



$A_D^+(v)$

For $v \in V(D)$, let $A_D^-(v)$ be the arcs with head v, $A_D^+(v)$ the arcs with tail v. For $S \subseteq V(D)$, let $A_D^-(S) = \bigcup_{v \in S} A_D^-(v)$ and $A_D^+(S) = \bigcup_{v \in S} A_D^+(v)$.

$A_D^-(S)$



$A_D^+(S)$

<u>Definition</u> Let (D, c) be a directed network, $s, t \in V(D)$. A function $f: A(D) \to \mathbb{R}$ is an <u>s-t-flow</u> of (D, c) if

$0 \le f(e) \le c(e)$ for all $e \in A(D)$ (capacity constraint) and

$\sum_{e \in A_D^-(v)} f(e) = \sum_{e \in A_D^+(v)} f(e)$ for all $v \in V(D) \setminus \{s, t\}$ (flow conservation constraint)

The <u>size</u> of the s-t-flow f is

$$|f| = \sum_{e \in A^+(s)} f(e) - \sum_{e \in A^-(s)} f(e).$$

(how much more leaves s than enters s)

An s-t-flow of (D, c) is a maximum s-t-flow of (D, c) if it has maximum size among all s-t-flows of (D, c).

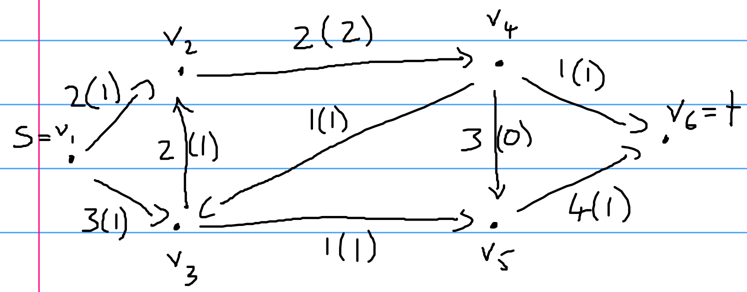
The extra amount of flow leaving s, which we defined to be the size of a flow, is equal to the extra amount of flow arriving at t:

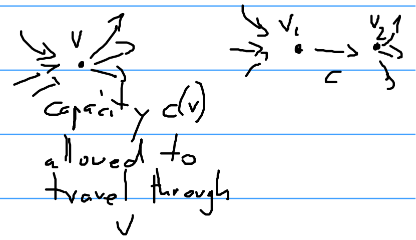
$$\left(\sum_{e\in A^+(s)} f(e) - \sum_{e\in A^-(s)} f(e)\right) + \left(\sum_{e\in A^+(t)} f(e) - \sum_{e\in A^-(t)} f(e)\right) =$$

$$= \sum_{v\in V(D)} \left(\sum_{e\in A^+(v)} f(e) - \sum_{e\in A^-(v)} f(e)\right) = 0$$

flow conservation for $v \in V(D)\setminus\{s,t\}$

$f(e)$ occurs twice in the sum, once with positive and once with negative sign



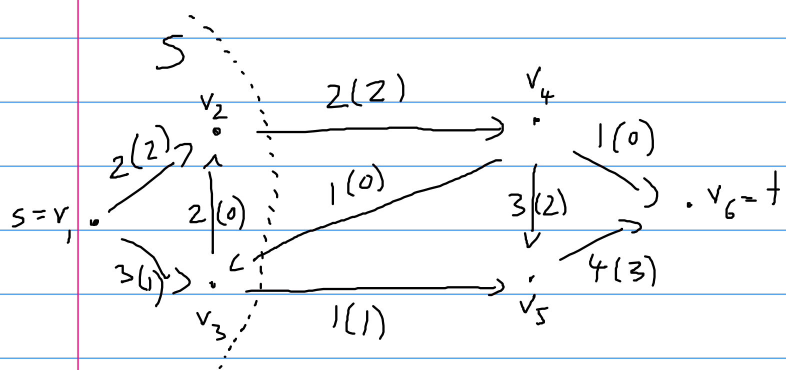


Let $c: A(D) \to \mathbb{R}$ be given by the numbers not in parentheses and $f: A(D) \to \mathbb{R}$ by the numbers in parentheses.

Then f is a v_1-v_6-flow:

- capacity constraints satisfied, e.g., $0 \le \underbrace{f(v_1v_2)}_{=1} \le \underbrace{c(v_1v_2)}_{=2}$
- flow conservation constraints satisfied, e.g.,

$$\underbrace{f(v_1v_2)}_{=1} + \underbrace{f(v_3v_2)}_{=1} = \underbrace{f(v_2v_4)}_{=2}$$



Let $g: A(D) \to \mathbb{R}$ be given by the numbers in parentheses.

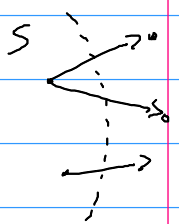
We can check that g is also a v_1-v_6-flow and $|g| = 3$. Thus $|g| > |f|$, so f is not a maximum v_1-v_6-flow.

6.2 Minimum Cuts

Definition Let (D, c) be a directed network, $s, t \in V(D)$. A set $S \subseteq V(D)$ is an *s-t-cut* of (D, c) is $s \in S$ and $t \notin S$. The *capacity* of an s-t-cut S is

$$C(S) = \sum_{e \in A_D^+(S) \cap A_D^-(V(D) \setminus S)} c(e)$$

arcs with tail in S and head outside S

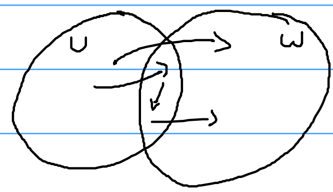


A minimum *s-t-cut* of (D, c) is an s-t-cut of (D, c) with minimum capacity among all s-t-cuts of (D, c).

Lemma Let (D, c) be a directed network, $s, t \in V(D)$. Let f be an s-t-flow of (D, c) and S an s-t-cut of (D, c). Then $|f| \leq C(S)$.

Proof. For $U, W \subseteq V(D)$, let $F(U, W)$ be the overall amount of flow on arcs with tail in U and head in W, i.e.,

$$F(U, W) = \sum_{e \in A_D^+(U) \cap A_D^-(W)} f(e)$$



$$|f| = \sum_{e \in A^+(s)} f(e) - \sum_{e \in A^-(s)} f(e) =$$

$$= \sum_{v \in S} \left(\sum_{e \in A^+(v)} f(e) - \sum_{e \in A^-(v)} f(e) \right) =$$

$s \in S$
$t \notin S$
flow conservation for $v \in S \setminus \{s\}$

$$= F(S, V(D)) - F(V(D), S) =$$

$$= F(S, V(D)\setminus S) + F(S,S) - F(V(D)\setminus S, S) - F(S,S)$$

$$= F(S, V(D)\setminus S) - F(V(D)\setminus S, S)$$

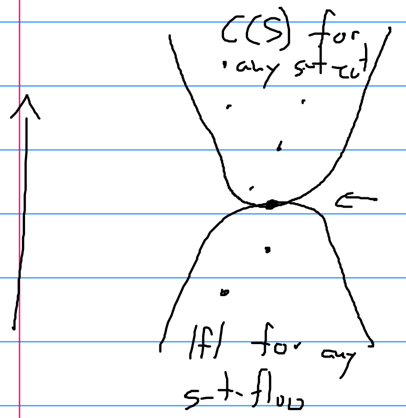
$$\le F(S, V(D)\setminus S) \le C(S)$$

$f(e) \geq 0$ for all $e \in A(D)$

$f(e) \leq c(e)$ for all $e \in A(D)$

$\square$

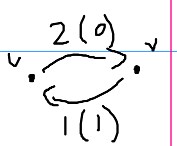
<u>Corollary</u> Let (D,c) be a directed network, $s,t \in V(D)$. Let f be an s-t-flow of (D,c) and S an s-t-cut of (D,c) such that $|f| = C(S)$. Then f is a maximum s-t-flow and S a minimum s-t-cut.



← $|f| = C(S)$ for some s-t-flow f and some s-t-cut C

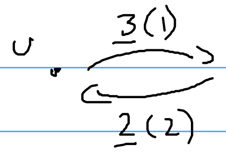
6.3 Residual Capacities and Augmenting Paths

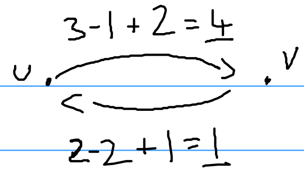
<u>Definition</u> Let (D,c) be a directed network, $s,t \in V(D)$. Let f be an s-t-flow of (D,c). Then the residual capacity $c_f(u,v)$ between $u \in V(D)$ and $v \in V(D)$ is



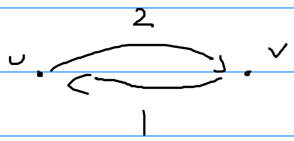
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) + f(v,u) & \text{if } uv \in A(D) \text{ and } vu \in A(D) \\ c(u,v) - f(u,v) & \text{if } uv \in A(D) \text{ and } vu \notin A(D) \\ f(v,u) & \text{if } uv \notin A(D) \text{ and } vu \in A(D) \\ 0 & \text{if } uv \notin A(D) \text{ and } vu \notin A(D) \end{cases}$$

capacity (flow)

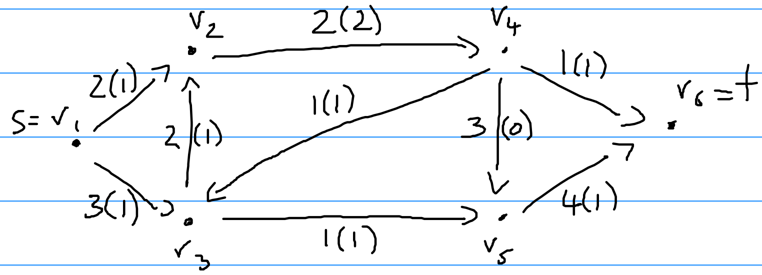




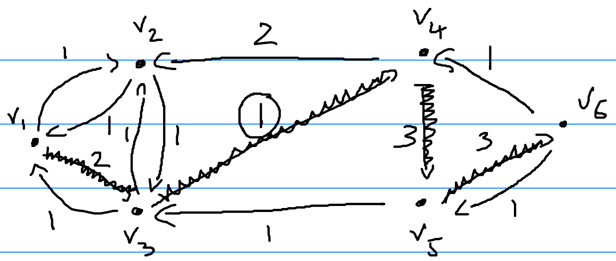
$$c_f(u,v) + c_f(v,u) = c(u,v) - \cancel{f(u,v)} + \cancel{f(v,u)} + c(v,u) - \cancel{f(v,u)} + \cancel{f(u,v)}$$
$$= c(u,v) + c(v,u)$$



Definition The residual network for (D,c) and f is the network (R, c_f) where $V(R) = V(D)$ and
$A(R) = \{uv : u,v \in V(R),\ c_f(u,v) > 0\}$



network
labels: capacity (flow)



residual network
labels: residual capacities

Definition Let (D,c) be a directed network, $s,t \in V(D)$. Let f be an s-t-flow of (D,c). Let $P = v_0 v_1 v_2 \dots v_m$ be a sequence of vertices in $V(D)$. Then P is an f-augmenting s-t-path if $v_0 = s$, $v_m = t$ and $c_f(v_{i-1}, v_i) > 0$ for all $i \in [m]$.
(in other words: it is a directed s-t-path in the residual network)

The residual capacity of P is $c_f(P) = \min_{i \in [m]} c_f(v_{i-1}, v_i)$.