MTH6105 – Algorithmic Graph Theory **Spring 2024**

Problem Sheet 8 **F. Fischer**

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

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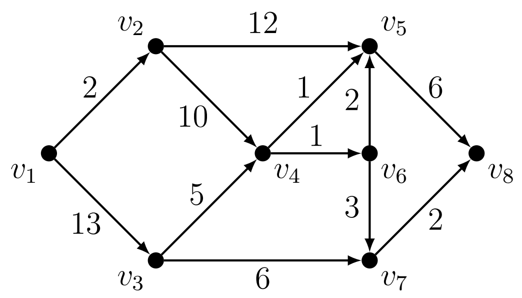
1. Consider the following directed network (D, c).



   (a) For each of the following functions from $A(D)$ to $\mathbb{R}$, determine if the function is a v_1–v_8-flow of (D, c). If a function is a v_1–v_8-flow, give its size.

   (i) $f_1(v_1v_2) = 2,\quad f_1(v_1v_3) = 2,\quad f_1(v_2v_4) = 0,\quad f_1(v_2v_5) = 2,$
   $f_1(v_3v_4) = 2,\quad f_1(v_3v_7) = 0,\quad f_1(v_4v_5) = 1,\quad f_1(v_4v_6) = 1,$
   $f_1(v_5v_8) = 2,\quad f_1(v_6v_5) = 1,\quad f_1(v_6v_7) = 2,\quad f_1(v_7v_8) = 2.$

   (ii) $f_2(v_1v_2) = 2,\quad f_2(v_1v_3) = 2,\quad f_2(v_2v_4) = 2,\quad f_2(v_2v_5) = 2,$
   $f_2(v_3v_4) = 0,\quad f_2(v_3v_7) = 2,\quad f_2(v_4v_5) = 1,\quad f_2(v_4v_6) = 1,$
   $f_2(v_5v_8) = 4,\quad f_2(v_6v_5) = 1,\quad f_2(v_6v_7) = 2,\quad f_2(v_7v_8) = 2.$

   (iii) $f_3(v_1v_2) = 2,\quad f_3(v_1v_3) = 4,\quad f_3(v_2v_4) = 0,\quad f_3(v_2v_5) = 2,$
   $f_3(v_3v_4) = 2,\quad f_3(v_3v_7) = 2,\quad f_3(v_4v_5) = 1,\quad f_3(v_4v_6) = 1,$
   $f_3(v_5v_8) = 4,\quad f_3(v_6v_5) = 1,\quad f_3(v_6v_7) = 0,\quad f_3(v_7v_8) = 2.$

   (iv) $f_4(v_1v_2) = 2,\quad f_4(v_1v_3) = 5,\quad f_4(v_2v_4) = 4,\quad f_4(v_2v_5) = 2,$
   $f_4(v_3v_4) = 3,\quad f_4(v_3v_7) = 2,\quad f_4(v_4v_5) = 1,\quad f_4(v_4v_6) = 2,$
   $f_4(v_5v_8) = 5,\quad f_4(v_6v_5) = 2,\quad f_4(v_6v_7) = 0,\quad f_4(v_7v_8) = 2.$

   (b) For each of the following subsets of $V(D)$, determine if the subset is a v_1–v_8-cut of (D, c). If a subset is a v_1–v_8-cut, give its capacity.

   (i) $S_1 = \{v_3, v_4, v_7\}$
   (ii) $S_2 = \{v_1, v_3, v_4, v_7\}$
   (iii) $S_3 = \{v_1, v_3, v_7, v_8\}$
   (iv) $S_4 = \{v_1, v_2, v_3, v_4, v_5, v_7\}$

   (c) Use your answers to Parts (a) and (b) to find a *maximum* v_1–v_8-flow and a *minimum* v_1–v_8-cut of (D, c).

Solution:

(a) (i) Function f_1 is not a v_1–v_8-flow because it violates the flow conservation constraint for v_5: $\sum_{e\in A_D^-(v_5)} f(e) = f(v_2v_5) + f(v_4v_5) + f(v_6v_5) = 4 \neq 2 = f(v_5v_8) = \sum_{e\in A_D^+(v_5)} f(e)$.

(ii) Function f_2 is not a v_1–v_8-flow because it violates the flow conservation constraint for v_2: $\sum_{e\in A_D^-(v_2)} f(e) = f(v_1v_2) = 2 \neq 4 = f(v_2v_4) + f(v_2v_5) = \sum_{e\in A_D^+(v_2)} f(e)$.

(iii) Function f_3 satisfies all capacity and flow conservation constraints and therefore is a v_1–v_8-flow. Its size is $|f_3| = \sum_{e\in A_D^+(V_1)} f(e) = f(v_1v_2) + f(v_1v_3) = 6$.

(iv) Function f_4 is not a v_1–v_8-flow because it violates the capacity constraint for v_4v_6: $f(v_4v_6) = 2 > 1 = c(v_4v_6)$.

(b) (i) Set S_1 is not a v_1–v_8-cut because $v_1 \notin S_1$.

(ii) Set S_2 is a v_1–v_8-cut because $v_1 \in S_2$ and $v_8 \in V(D) \setminus S_2$. Its capacity is $C(S_2) = \sum_{e\in A_D^+(S)\cap A_D^-(V(D)\setminus S)} c(e) = c(v_1v_2) + c(v_4v_5) + c(v_4v_6) + c(v_7v_8) = 6$.

(iii) Set S_3 is not a v_1–v_8-cut because $v_8 \notin V(D) \setminus S_3$.

(iv) Set S_4 is a v_1–v_8-cut because $v_1 \in S_2$ and $v_8 \in V(D) \setminus S_2$. Its capacity is $C(S_4) = \sum_{e\in A_D^+(S)\cap A_D^-(V(D)\setminus S)} c(e) = c(v_4v_6) + c(v_5v_8) + c(v_7v_8) = 9$.

(c) The v_1–v_8-flow f_3 and the v_1–v_8-cut S_2 satisfy $|f_3| = C(S_2)$. Thus, by Corollary 6.5 in the notes, f_3 is a maximum v_1–v_8-flow and S_2 a minimum v_1–v_8-cut.

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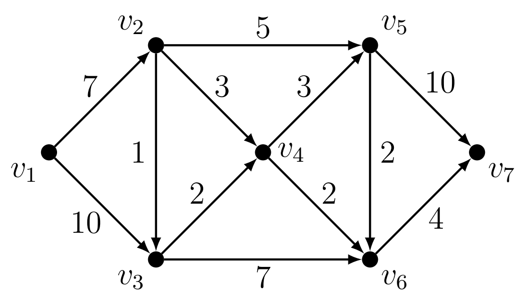
2. Consider the following directed network (D, c).



(a) Let g_1 be the v_1–v_7-flows of (D, c) with

$$
\begin{array}{llll}
g_1(v_1v_2) = 7, & g_1(v_1v_3) = 1, & g_1(v_2v_3) = 1, & g_1(v_2v_4) = 3, \\
g_1(v_2v_5) = 3, & g_1(v_3v_4) = 2, & g_1(v_3v_6) = 0, & g_1(v_4v_5) = 3, \\
g_1(v_4v_6) = 2, & g_1(v_5v_6) = 2, & g_1(v_5v_7) = 4, & g_1(v_6v_7) = 4.
\end{array}
$$

(i) Draw the residual network for (D, c) and g_1.

(ii) Give two distinct g_1-augmenting v_1–v_7-paths, along with their residual capacities.

(b) Let g_2 be the v_1–v_7-flows of (D, c) with

$$\begin{aligned}
&g_2(v_1v_2) = 7, && g_2(v_1v_3) = 4, && g_2(v_2v_3) = 0, && g_2(v_2v_4) = 2,\\
&g_2(v_2v_5) = 5, && g_2(v_3v_4) = 2, && g_2(v_3v_6) = 2, && g_2(v_4v_5) = 2,\\
&g_2(v_4v_6) = 2, && g_2(v_5v_6) = 0, && g_2(v_5v_7) = 7, && g_2(v_6v_7) = 4.
\end{aligned}$$

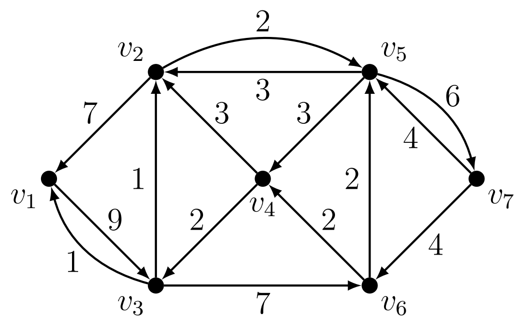
(i) Draw the residual network for (D, c) and g_2.

(ii) Give a g_2-augmenting v_1–v_7-path.

(iii) Show that g_2 is not a maximum v_1–v_7-flow of (D, c), by giving a v_1–v_7-flow g_3 with $|g_3| > |g_2|$.

Solution:

(a) (i) The residual network looks as follows.



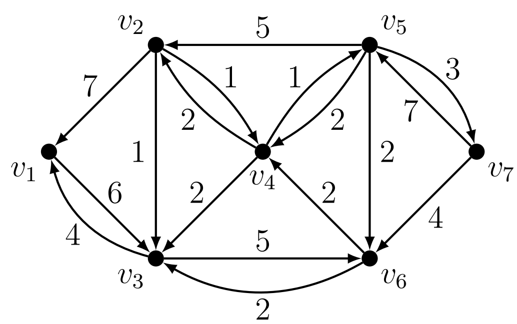
(ii) The g_1-augmenting v_1–v_7-paths include v_1, v_3, v_2, v_5, v_7, which has residual capacity

$$\min\{c_{g_1}(v_1v_3), c_{g_1}(v_3v_2), c_{g_1}(v_2v_5), c_{g_1}(v_5v_7)\} = 1,$$

and v_1, v_3, v_6, v_5, v_7, which has residual capacity

$$\min\{c_{g_1}(v_1v_3), c_{g_1}(v_3v_6), c_{g_1}(v_6v_5), c_{g_1}(v_5v_7)\} = 2.$$

(b) (i) The residual network looks as follows.



(ii) The unique g_2-augmenting v_1–v_7-path is $P = v_1, v_3, v_6, v_4, v_5, v_7$.