MTH6105 – Algorithmic Graph Theory **Spring 2024**

Problem Sheet 8 **F. Fischer**

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

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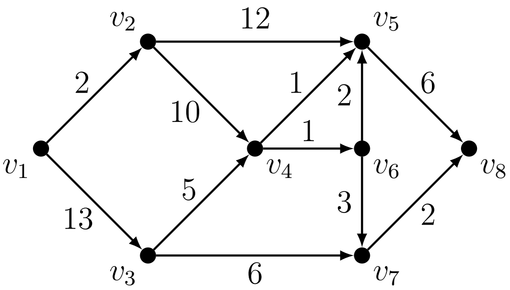
1. Consider the following directed network (D, c).



(a) For each of the following functions from $A(D)$ to $\mathbb{R}$, determine if the function is a v_1–v_8-flow of (D, c). If a function is a v_1–v_8-flow, give its size.

(i)
$$f_1(v_1v_2) = 2, \quad f_1(v_1v_3) = 2, \quad f_1(v_2v_4) = 0, \quad f_1(v_2v_5) = 2,$$
$$f_1(v_3v_4) = 2, \quad f_1(v_3v_7) = 0, \quad f_1(v_4v_5) = 1, \quad f_1(v_4v_6) = 1,$$
$$f_1(v_5v_8) = 2, \quad f_1(v_6v_5) = 1, \quad f_1(v_6v_7) = 2, \quad f_1(v_7v_8) = 2.$$

(ii)
$$f_2(v_1v_2) = 2, \quad f_2(v_1v_3) = 2, \quad f_2(v_2v_4) = 2, \quad f_2(v_2v_5) = 2,$$
$$f_2(v_3v_4) = 0, \quad f_2(v_3v_7) = 2, \quad f_2(v_4v_5) = 1, \quad f_2(v_4v_6) = 1,$$
$$f_2(v_5v_8) = 4, \quad f_2(v_6v_5) = 1, \quad f_2(v_6v_7) = 2, \quad f_2(v_7v_8) = 2.$$

(iii)
$$f_3(v_1v_2) = 2, \quad f_3(v_1v_3) = 4, \quad f_3(v_2v_4) = 0, \quad f_3(v_2v_5) = 2,$$
$$f_3(v_3v_4) = 2, \quad f_3(v_3v_7) = 2, \quad f_3(v_4v_5) = 1, \quad f_3(v_4v_6) = 1,$$
$$f_3(v_5v_8) = 4, \quad f_3(v_6v_5) = 1, \quad f_3(v_6v_7) = 0, \quad f_3(v_7v_8) = 2.$$

(iv)
$$f_4(v_1v_2) = 2, \quad f_4(v_1v_3) = 5, \quad f_4(v_2v_4) = 4, \quad f_4(v_2v_5) = 2,$$
$$f_4(v_3v_4) = 3, \quad f_4(v_3v_7) = 2, \quad f_4(v_4v_5) = 1, \quad f_4(v_4v_6) = 2,$$
$$f_4(v_5v_8) = 5, \quad f_4(v_6v_5) = 2, \quad f_4(v_6v_7) = 0, \quad f_4(v_7v_8) = 2.$$

(b) For each of the following subsets of $V(D)$, determine if the subset is a v_1–v_8-cut of (D, c). If a subset is a v_1–v_8-cut, give its capacity.

(i) $S_1 = \{v_3, v_4, v_7\}$

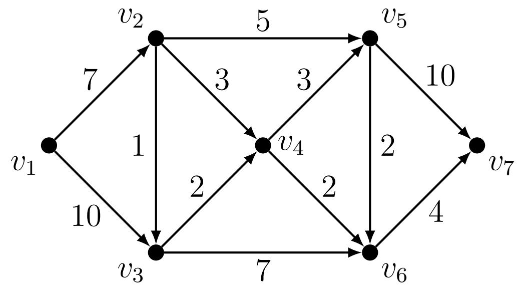
(ii) $S_2 = \{v_1, v_3, v_4, v_7\}$

(iii) $S_3 = \{v_1, v_3, v_7, v_8\}$

(iv) $S_4 = \{v_1, v_2, v_3, v_4, v_5, v_7\}$

(c) Use your answers to Parts (a) and (b) to find a *maximum* v_1–v_8-flow and a *minimum* v_1–v_8-cut of (D, c).

2. Consider the following directed network (D, c).



(a) Let g_1 be the v_1–v_7-flows of (D, c) with

$$\begin{aligned} &g_1(v_1v_2) = 7, && g_1(v_1v_3) = 1, && g_1(v_2v_3) = 1, && g_1(v_2v_4) = 3, \\ &g_1(v_2v_5) = 3, && g_1(v_3v_4) = 2, && g_1(v_3v_6) = 0, && g_1(v_4v_5) = 3, \\ &g_1(v_4v_6) = 2, && g_1(v_5v_6) = 2, && g_1(v_5v_7) = 4, && g_1(v_6v_7) = 4. \end{aligned}$$

(i) Draw the residual network for (D, c) and g_1.

(ii) Give two distinct g_1-augmenting v_1–v_7-paths, along with their residual capacities.

(b) Let g_2 be the v_1–v_7-flows of (D, c) with

$$\begin{aligned} &g_2(v_1v_2) = 7, && g_2(v_1v_3) = 4, && g_2(v_2v_3) = 0, && g_2(v_2v_4) = 2, \\ &g_2(v_2v_5) = 5, && g_2(v_3v_4) = 2, && g_2(v_3v_6) = 2, && g_2(v_4v_5) = 2, \\ &g_2(v_4v_6) = 2, && g_2(v_5v_6) = 0, && g_2(v_5v_7) = 7, && g_2(v_6v_7) = 4. \end{aligned}$$

(i) Draw the residual network for (D, c) and g_2.

(ii) Give a g_2-augmenting v_1–v_7-path.

(iii) Show that g_2 is not a maximum v_1–v_7-flow of (D, c), by giving a v_1–v_7-flow g_3 with $|g_3| > |g_2|$.