Week 6

Theorem 9.1

Suppose the price $S(t)$ of a share ~~at~~ satisfies the relation

$$\tilde{E}\left(S(T)\right) = S_0 e^{rT}$$

where r is the interest rate compounded continuously

$$S_0 = S(0) = S$$

then $$C - P = S_0 - e^{-rT} K$$

Proof: $C = e^{-rT}\tilde{E}(S(T)-K)^+$ and $P = e^{-rT}\tilde{E}\left(K - S(T)\right)^+$ → Week 3-4 the 3 egs

$$C - P = e^{-rT}\left[\tilde{E}(S(T)-K)^+ - \tilde{E}(K-S(T))^+\right]$$

$$= e^{-rT}\tilde{E}\left[(S(T)-K)^+ - (K-S(T))^+\right]$$

$E(X) - E(Y) = E(X-Y)$

$$= e^{-rT}\tilde{E}[S(T)-K]$$

$$= e^{-rT}[\tilde{E}(S(T)) - K]$$

$$= e^{-rT}[S_0 e^{rT} - K]$$

$$= S_0 - e^{-rT}K. \quad \boxtimes$$

$x^+ - (-x)^+ = x$
for all real number

$\tilde{E}(S(T)) = S_0 e^{rT}$

Call-put parity: $c - p = S_0 - e^{-rT}K$

No dividend

With dividend: $c - p = e^{-qT}S_0 - e^{-rT}K$.

Indices

Theorem 10.1

$I(t)$: $S_1(t), S_2(t), \cdots, S_n(t)$. $\leftarrow$ prices

$w_1, \quad w_2, \quad \cdots, \quad w_n \quad \leftarrow$ weights

$C_I(k,t)$: price of the call option on the index $I(t)$.

$C_j(k,t)$: price of the call options on the stock with the price $S_j(t)$, $j=1,\cdots,n$.

Then

$$C_I(k,t) \le \sum_{j=1}^{n} w_j C_j(k,t)$$

Proof:

$$C_I(k,t) = e^{-rt}\tilde{E}\left(I(t)-k\right)^+$$

$$C_j(k,t) = e^{-rt}\tilde{E}\left(S_j(t)-k\right)^+$$

$$I(t)-k = \sum_{j=1}^{n} w_j S_j(t) - k \overset{*}{=} \sum_{j=1}^{n} w_j\left(S_j(t)-k\right),$$

~~S(t)-k~~

$$k = k\sum_{j=1}^{n} w_j = \sum_{j=1}^{n}(w_j k) \quad (*)$$

$$\left(I(t)-K\right)^{+} = \left(\sum_{j=1}^{n} w_j \left(S_j(t)-K\right)\right)^{+}$$

$$\overset{**}{\leq} \sum_{j=1}^{n} \left[w_j \left(S_j(\tau)-K\right)\right]^{+}$$

Fact: $\left(\sum_{j=1}^{n} x_j\right)^{+} \leq \sum_{j=1}^{n} (x_j)^{+}$ $**$

$$\left(x_1+\cdots+x_n\right)^{+} \leq (x_1)^{+}+(x_2)^{+}+\cdots+(x_n)^{+}$$

$$= \sum_{j=1}^{n} w_j \left(S_j(t)-K\right)^{+}$$

$$e^{-rt}\tilde{E}\left(I(t)-K\right)^{+} \leq \left[\sum_{j=1}^{n} w_j E\left(S_j(t)-K\right)^{+}\right]\cdot e^{-rt}$$

$$= \sum_{j=1}^{n} w_j \underbrace{e^{-rt} E\left(S_j(t)-K\right)^{+}}$$

W6 ④

$$C_I(k,t) \le \sum_{j=1}^{n} w_j C_j(k,t)$$ $\boxed{\square}$

Volatility

① Historical volatility $\qquad s^2 \qquad$ historical data

② Implied volatility

B-S $\qquad$ In practice

σ constant

$\sigma \leftarrow$ Call price $\leftarrow$ B-S

$\uparrow$ unrealistic assumptions

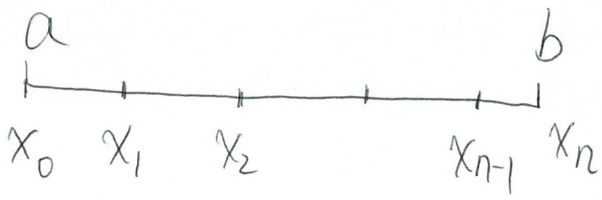
conflict $\qquad \sigma(t)$ or $\sigma(K)$

$\Rightarrow$ Volatility smile

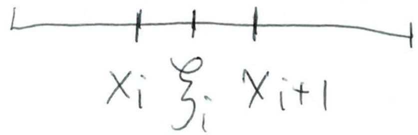
12. Stochastic Calculus

12.1 The def of the integral

$$\int_a^b f(x)\,dx \qquad f: [a,b] \to \mathbb{R}$$



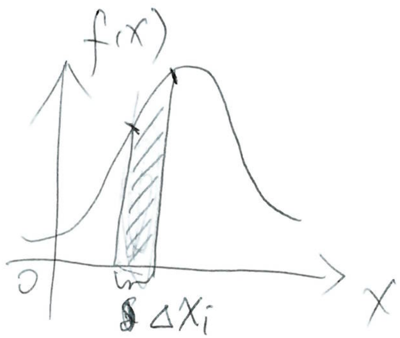
1. $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$



2. Integral sum $\displaystyle\sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i$, $\Delta x_i = x_{i+1} - x_i$

3. Set $\displaystyle\delta = \max_{0 \le i \le n-1} \Delta x_i$ $\qquad \xi_i \in [x_i, x_{i+1})$

4. $$\int_a^b f(x)\,dx = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(\xi_i) \cdot \Delta x_i$$



Theorem 12.1 existence of lim

If $f(x)$ is a continuous function on $[a, b]$
then the limit $\lim\limits_{\delta \to 0} \sum_{i=1}^{n-1} f(\xi_i) \Delta x_i$ exists.

Properties of normal r.vs

If $Z_1, Z_2, \cdots Z_n$ are independent Normal r.vs.

$Z_i \sim N(\mu_i, \sigma_i^2)$, then

$$\sum_{i=1}^{n} Z_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right) \qquad (5)$$

Properties of Wiener process $W(t) \equiv W_t$

1. $W(0) = 0$, 2. $W(t+s) - W(t) \sim N(0, s)$, if $s > 0$
3. $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t$ $\quad [0, t]$

$\Delta W_i = W(t_{i+1}) - W(t_i)$, $i = 0, 1, \cdots, n-1$, and $\Delta t_i = t_{i+1} - t_i$

Then ΔW_i are independent normal r.v.s

$\Delta W_i \sim N(0, \Delta t_i)$

Goals: $\int_0^t f(x)\,dx \rightarrow \int_0^t f(s)\,dW_s$ $\quad W_s = W(S)$

$\int_0^t f(W_s)\,dW_s$

(a) $\int_0^t f(s)\,dW_s$

(b) $\int_0^t f(s)\,dW_s \sim$? distribution

(c) $\int_0^t f(W_s)\,dW_s$

(a) Stochastic integral $\int_0^t f(s)\,dW_s$

Def. 12.1

Let $t_0 = 0 < t_1 < t_2 < \cdots < t_n = t$ $\quad [0, t]$ $\quad \Delta t_i = t_{i+1} - t_i$

$\delta = \max_i \Delta t_i$.

Then $$\int_0^t f(s)\,dW_s = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(t_i)\,\underline{\Delta W_i} \qquad (7)$$

if this limit exists.

Theorem 12.2 existence

If $f(x)$ is differentiable and $f'(x)$ is a continuous function then the limit in (7) exists.

Example 1.

$f(x) = c$, then

$$\int_0^t f(s)\,dW_s = \int_0^t c\,dW_s \overset{(7)}{=} \lim_{\delta\to 0} \sum_{i=0}^{n-1} c\,\Delta W_i = c \underbrace{\lim_{\delta\to 0} \sum_{i=0}^{n-1} \Delta W_i}$$

$\Delta W_i = W(t_{i+1}) - W(t_i)$

$$\lim_{\delta\to 0} \sum_{i=0}^{n-1} \Delta W_i = \underbrace{[W(t_1) - W(t_0)]}_{i=0} + \underbrace{[W(t_2) - \cancel{W(t_1)}]}_{i=1} + \cdots + \underbrace{[W(t_n) - W(t_{n-1})]}_{i=n-1}$$

$= W(t_n) - W(t_0) = W(t) = W_t$

$$\underset{\substack{\| \\ t}}{W(\cancel{t}\,t_n)} - \underset{\substack{\|_0 \\ =0}}{W(\cancel{t}\,t_0)} = W(\cancel{\ \ }\ t)$$

$$\int_0^t c\, dW_s = c \lim_{\delta \to 0}^{\cancel{n}} W_t = c W_t$$

Example 2.

$$f(x) = \begin{cases} 1 & 0 \le x < 1.5 \\ -1 & 1.5 \le x < 2 \end{cases}$$

$$\int_0^2 f(s)\, dW_s = \int_0^{1.5} f(s)\, dW_s + \int_{1.5}^2 f(s)\, dW_s$$

$$= \int_0^{1.5} dW_s - \int_{1.5}^2 dW_s$$

$$= W(1.5) - W(0) - [W(2) - W(1.5)]$$

$$= 2W(1.5) - W(2)$$

$$W(1.5) + W(1.5) - W(2)$$

$$W(1.5) - W(0.5)$$

According Eg 1.

$$\int_0^t c\, dW_s = c W(t)$$

$$\boxed{\int_a^b dW_s = W(b) - W(a)}$$

CW Q4 (a)

(b) Distribution of r.v. $\int_0^t f(s)\,dW_s = \lim_{\delta \to 0} \sum_{i=0}^{n-1} \underbrace{f(t_i)\,\Delta W_i}_{}$ (7), with ΔW_i a r.v

Theorem 12.3

$$\int_0^t f(s)\,dW_s \sim N\left(0, \int_0^t (f(s))^2\,ds\right).$$

Proof: $\int_0^t f(s)\,dW_s \simeq \sum_{i=0}^{n-1} \underbrace{f(t_i)\,\Delta W_i}$

$\underbrace{f(t_i)}_{\text{not r.v.}} \Delta W_i \sim N(0, f^2(t_i)\,\Delta t_i)$

$$\sum_{i=0}^{n-1} f(t_i)\,\Delta W_i \sim \sum_{i=0}^{n-1} N(0, f^2(t_i)\,\Delta t_i)$$

$$\overset{(5)}{=} N\left(0, \sum_{i=0}^{n-1} f^2(t_i)\,\Delta t_i\right)$$

$$\lim_{\max_i \Delta t_i \to 0} \sum_{i=0}^{n-1} f^2(t_i)\,\Delta t_i = \int_0^t f^2(s)\,ds \quad \blacksquare$$

$\Delta W_i = W(t_{i+1}) - W(t_i) \sim N(0, \Delta t_i)$

$c\,\Delta W_i \sim N(0, c^2 \Delta t_i)$

$\downarrow$

$f(t_i)$

$$\sum_{i=1}^{n} Z_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right) \quad (5)$$

(C) $\int_0^t f(W_s)\, dW_s \leftarrow$ r.v

$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$, $\delta = \max_{0 \le i \le n-1} (t_{i+1} - t_i)$

Def 12.2

f : $\to$ be a function

If limit $\lim_{\delta \to 0} f(W(t_i)) \Delta W_i$ exists,

then $\int_a^b f(W_t)\, dW_t = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(\underset{\substack{\| \\ W_i}}{W(t_i)}) \Delta W_i$ (8)

Theorem 12.4 existence

If f has a bounded continuous derivative $f'(x)$,

then limit in (8) exists.

Theorem 12.5

$$E\left(\int_a^b f(W_t)\,dW_t\right) = 0 \qquad (9)$$

$$\mathrm{Var}\left(\int_a^b f(W_t)\,dW_t\right) = \int_a^b E\left((f(W_t))^2\right) dt \qquad (10)$$

Explanation:

Notations: $W_i = W(t_i)$, $f_i = f(W_i)$, $\Delta W_i = W(t_{i+1}) - W(t_i)$ i start 0

$W_i \perp \Delta W_i$ $W(t_i) \perp W(t_{i+1}) - W(t_i)$ $n-1$ Independent increments

$\Rightarrow f_i = f(W(t_i)) \perp \Delta W_i$

$$E(f_i \Delta W_i) \overset{(8)\ \text{def}}{=} E(f_i) \times \underbrace{E(\Delta W_i)} = 0 \quad (*) \qquad E(\Delta W_i) = 0$$

$$E\left(\sum_{i=0}^{n-1} f_i \Delta W_i\right) = \sum_{i=0}^{n-1} \underbrace{E(f_i \Delta W_i)} = 0$$

$$E\left(\int_a^b f(W_t)\,dW_t\right) = \lim_{\delta\to 0} E\Big(\overbrace{\sum_{i=1}^{n-1} f(W(t_i))\Delta W_i}^{=0}\Big) = 0$$

$$\mathrm{cov}(f_i \Delta W_i,\ f_j \Delta W_j) = E[f_i \Delta W_i \times f_j \Delta W_j] - \underbrace{E[f_i \Delta W_i]}_{(*):\ 0} \cdot \underbrace{E[f_j \Delta W_j]}_{0}$$

$$= E[f_i \Delta W_i \times f_j \Delta W_j]$$

$$= E[f_i \Delta W_i f_j] \cdot \underbrace{E[\Delta W_j]}_{} \qquad \Delta W_j \perp f_i \Delta W_i f_j$$

$$= 0$$

$$\mathrm{Var}\left[\sum_{i=0}^{n-1} f(W_i)\Delta W_i\right] = \sum_{i=0}^{n-1} \mathrm{Var}(f_i \Delta W_i) \qquad \because \mathrm{cov} = 0$$

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

$$= \sum_{i=1}^{n-1}\left[E(f_i^2 \Delta W_i^2) - [\underbrace{E(f_i \Delta W_i)}_{=0}]^2\right]$$

$$= \sum_{i=1}^{n-1} E(f_i^2 \Delta W_i^2) = \sum_{i=0}^{n-1} E(f_i^2) \times \underbrace{E(\Delta W_i^2)}_{}$$

$$= \sum_{i=0}^{n-1} E(f_i^2) \times \Delta t_i$$

$$E(\Delta W_i^2) = E\left[(W(t_{i+1}) - W(t_i))^2\right] \quad \Delta t_i$$

$$\cancel{= E(W^2(t_{i+1})) + E(W^2(t_i))}$$

$$\cancel{-2\, E(W(t_{i+1}) \cdot W(t_i))}$$

$$\cancel{= 0 \ \text{independence}}$$

W6 (14)

Lemma 1.2

$$E\left(W_t^{2j}\right) = \frac{(2j)!}{j!\,2^j} t^j$$

$$E\left(W_t^{2j-1}\right) = 0$$