# Week 6 Copula

Review: marginal / individual distribution

$F_X(x)$, $F_Y(y)$

joint distribution

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$F_{X_1,\cdots,X_d}(x_1, x_2, \cdots, x_d) = P(X_1 \le x_1, X_2 \le x_2, \cdots, X_d \le x_d)$$

| Extreme Value Theory | Copula |
|---|---|
| $X$<br>extreme : tail<br>tail weight<br><br>upper tail | $X_1, X_2, \cdots, X_d$<br>large losses occur together<br>tail dependence<br>upper tail + lower tail |

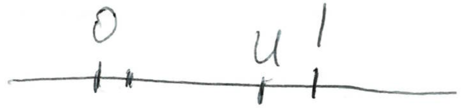
upper tail dependence

$$\lambda_U = \lim_{u \to 1^-} P\left(X > F_X^{-1}(u) \mid Y > F_Y^{-1}(u)\right)$$

$\downarrow$

Prob $\quad 0 \le \lambda_U \le 1$

$$\lambda_U = \lim_{u \to 1^-} P\left(\underbrace{X > x_0}_{\text{high value of } X} \mid \underbrace{Y > y_0}_{\text{high value of } Y}\right)$$

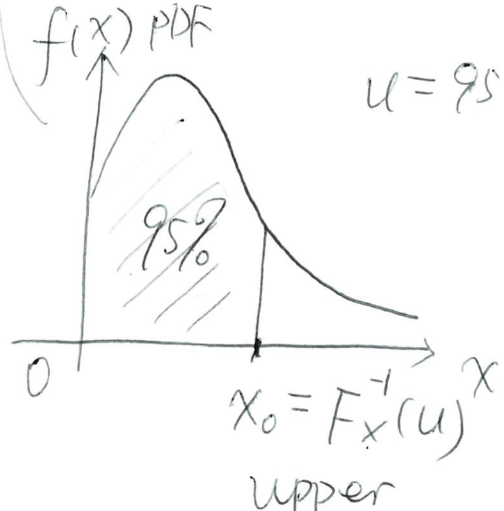



$u$ : percentile

$u \to 1$

$F_X^{-1}(u) = x_0$

$F_Y^{-1}(u) = y_0$

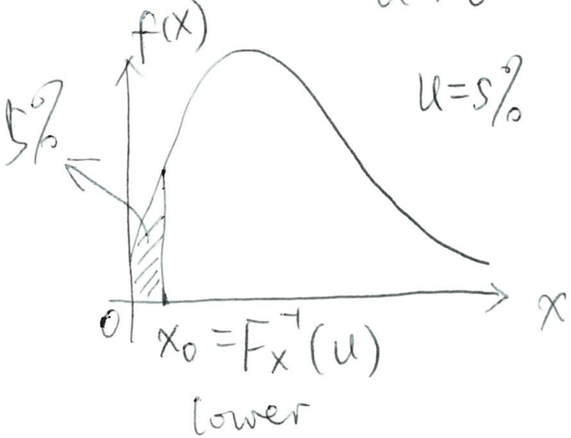



$u = 95\%$

Lower tail dependence

$$\lambda_L = \lim_{u \to 0^+} P\left(\underbrace{X \le F_X^{-1}(u)}_{\text{low value of } X} \mid \underbrace{Y \le F_Y^{-1}(u)}_{\text{low value of } Y}\right)$$

$u = 5\%$

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Def of Copula:

A copula: a function that expresses a multivariate CDF in terms of individual marginal CDFs

input: individual marginal CDFs

output: joint CDF

$X, Y, \quad C_{XY}$

$$C_{XY}[\underbrace{F_X(x)}_{u}, \underbrace{F_Y(y)}_{v}] = F_{X,Y}(x, y) = P(X \le x, Y \le y)$$

More compact form: $\quad u = F_X(x), \quad v = F_Y(y)$

$$= C(u, v) = F_{X,Y}(x, y)$$

$$C(u_1, u_2, \cdots, u_d) = F_{X_1, X_2, \cdots, X_d}(x_1, x_2, \cdots, x_d) \qquad [0, 1]$$

where $u_i = F_{X_i}(x_i)$

Q: Slide 11

Explain in words, the meaning of the following copula expression:

$C[u, v, w]$

A: $C[u, v, w] = F_{X,Y,Z}(x, y, z) = P(X \leq x, Y \leq y, Z \leq z)$

$u = F_X(x)$, $v = F_Y(y)$, $w = F_Z(z)$

This gives the probability that r.v 1 (X) is in the bottom $u$ percentile,

and r.v. 2 (Y) is in the bottom $v$ percentile,

and r.v. 3 (Z) is in the bottom $w$ percentile.

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Sklar's Theorem — existence of Copula

Let $F$ be a joint CDF with marginal CDFs $F_1, F_2, \cdots, F_d$.

Then there exists a Copula, $C$, s.t. for all $x_1, \cdots, x_d \in [-\infty, \infty]$

$$F(x_1, x_2, \cdots, x_d) = C[F_1(x_1), \cdots, F_d(x_d)]$$

In the case where variables ~~are cont~~ have a continuous distribution, the copula is unique.

The converse also holds:
If $C$ is a copula and $F_1, \dots, F_d$ are univariate CDFs, then the function $F$ defined above is a joint CDF with marginal CDF $F_1, \dots, F_d$.

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Tail dependence and survival copula (Fact)

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u,u)}{u} \rightarrow \text{lower tail}$$

Survival Copula: upper tail

$$\bar{F}(x,y) = P(X > x, Y > y) = \bar{C}[\bar{F}_X(x), \bar{F}_Y(y)]$$

$$\underset{1-u}{\bar{F}(x)} = 1 - \underset{u}{F_X(x)}, \quad \underset{1-v}{\bar{F}_Y(y)} = 1 - \underset{v}{F_Y(y)}$$

Relationship $C(u, v)$ v.s $\bar{C}(\dots, \dots)$


$$P(X \le x \overset{\text{or}}{\cup} Y \le y) = P(X \le x) + P(Y \le y) - P(X \le x, Y \le y)$$

$$1 - P(X > x, Y > y) = P(X \le x) + P(Y \le y) - P(X \le x, Y \le y)$$

$$P(X > x, Y > y) = 1 - P(X \le x) - P(Y \le y) + P(X \le x, Y \le y)$$

$$\boxed{\bar{C}[1-u, 1-v] = 1 - u - v + C(u, v)} \quad (*)$$

$$\bar{C}[\bar{F}_X(x), \bar{F}_Y(y)]$$

Fact: $\lambda_u = \lim_{u \to 1^-} P[X > F_X^{-1}(u) \mid Y > F_Y^{-1}(u)] = \lim_{u' \to 0^+} \frac{\bar{C}(u', u')}{u'}$

$$\overset{*}{=} \lim_{u \to 1^-} \left( \frac{1 - 2u + C(u, u)}{1 - u} \right) \qquad u' = 1 - u$$

# Types of Copula functions

3 main families of Copulas:

① Fundamental Copulas: Independence, perfect positive interdependence, perfect negative interdependence

② Explicit Copulas

③ Implicit Copulas

## ① Fundamental Copulas

### 1.1 Independence copula (product copula)

$$C(u, v) = uv$$

$$F_{X,Y}(x, y) = C[\overset{u}{F_X(x)}, \overset{v}{F_Y(y)}] = \underset{\overset{\parallel}{u}}{F_X(x)}\ \underset{\overset{\parallel}{v}}{F_Y(y)}$$

$\Updownarrow$

$$P(X \le x,\ Y \le y) = P(X \le x)\, P(Y \le y) \quad (*)$$

$\Leftarrow$ Property of 2 independent r.v.s $X, Y$

## 1.2 Co-monotonic copula (minimum Copula)

$$C(u, v) = \min(u, v) \qquad u = F_X(x), \quad v = F_Y(y)$$

$$C[F_X(x), F_Y(y)] = \min(F_X(x), F_Y(y))$$

$$P(X \le x, Y \le y) = \min(P(X \le x), P(Y \le y))$$

$$\min C[F_{X_1}(x_1), \ldots, F_{X_d}(x_d)] = \min[F_{X_1}(x_1), \ldots, F_{X_d}(x_d)]$$

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imagine $X = Y$


if $x < y$, $\quad P(X \le x, Y \le y) = P(X \le x, X \le y) = \underbrace{P(X \le x)}_{\min} \le P(X \le y)$

if $x \ge y$

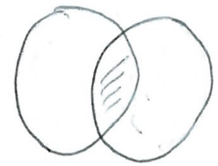
$$P(X \le y)$$

1.3 Counter-monotonic Copula (maximum Copula)

$$C(u, v) = \max(u+v-1, 0)$$

$$C[F_X(x), F_Y(y)] = \max[F_X(x)+F_Y(y)-1, 0]$$

$$P(X \le x, \cancel{P(Y} Y \le y) = \max(P(X \le x) + P(Y \le y) - 1, 0)$$



2. Explicit Copulas — Archimedean Copulas

↓

Simple closed form expressions

Archimedean copulous are a subset of explicit copulas

$$C[u, v] = \psi^{[-1]}(\psi(u) + \psi(v))$$

$\psi^{[-1]}$: pseudo-inverse function

$$\psi^{[-1]}(x) = \begin{cases} \psi^{-1}(x) & \text{if } 0 \le x \le \psi(0) \\ 0 & \text{if } \psi(0) < x \le \infty \end{cases}$$

$\psi^{(-1)}(x)$ : ordinary inverse function $\quad x = \psi(y)$

express $y$ in terms of $x$

$$\boxed{\text{If } \psi(0) = \infty, \ \cancel{\text{the}} \ \psi^{[-1]} = \psi^{(-1)}}$$

↑ pseudo-inverse

↖ ordinary inverse

Archimedean Copulas :

- The Gumbel Copula
- The Clayton copula
- The Frank Copula

2.1 The Gumbel Copula (Gumbel-Hougaard Copula)

$$C(u,v) = \exp\left\{ -\left( (-\ln u)^{\alpha} + (-\ln v)^{\alpha} \right)^{\frac{1}{\alpha}} \right\} \text{ for } \alpha \geq 1$$

Generator function: $\psi(t) = (-\ln t)^{\alpha}$ where $1 \leq \alpha < \infty$

- There is upper tail dependence $\leftarrow \alpha$
- There is no lower tail dependence
- $\alpha > 1$: positive upper tail dependence.

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Q: slide 24

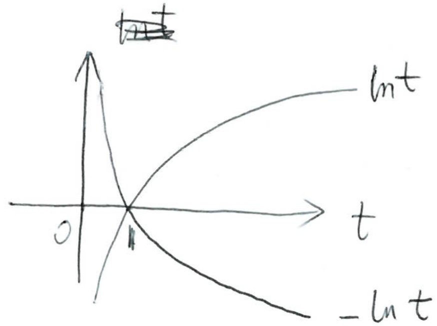
Derive ~~the~~ an expression for the Gumbel Copula for the case where there a 3 variables.
The generator function: $\psi(t) = (-\ln t)^{\alpha}$ where $1 \leq \alpha < \infty$

~~As~~ A: • Gumbel Copula is an example of Archimedean copulas.

• $C(u, v, w) = \psi^{[-1]}(\psi(u) + \psi(v) + \psi(w))$

• ☆ Find $\psi^{[-1]}$, the pseudo-inverse ~~f~~ generator function

check $\psi(0) = \cancel{\lim_{t \to 0} (-\ln(t))}$




$$\lim_{t \to 0} (-\ln t)^{\alpha} = \infty$$

So $\psi^{[-1]} = \psi^{(-1)}$

• $y = \psi^{(-1)}(x)$, then $\psi(y) = x$

So $(-\ln y)^{\alpha} = x$

$-\ln y = x^{\frac{1}{\alpha}}$ $\quad \ln y = -x^{\frac{1}{\alpha}}$

$y = \exp(-x^{\frac{1}{\alpha}})$

$\boxed{\psi^{(-1)}(x)} = \exp(-x^{\frac{1}{\alpha}})$ (**)

Generator func Gumbel Copula:
$\psi(t) = (-\ln t)^{\alpha}$

- $C(u,v,w) = \psi^{[-1]}\left(\psi(u)+\psi(v)+\psi(w)\right)$

$$= \psi^{[-1]}\left((-\ln u)^{\alpha} + (-\ln v)^{\alpha} + (-\ln w)^{\alpha}\right)$$

$$= \psi^{(-1)}\left((-\ln u)^{\alpha} + (-\ln v)^{\alpha} + (-\ln w)^{\alpha}\right) \qquad \psi(0)=\infty$$

$$\overset{**}{=} \exp\left\{-\left[(-\ln u)^{\alpha} + (-\ln v)^{\alpha} + (-\ln w)^{\alpha}\right]^{\frac{1}{\alpha}}\right\}$$

## 2.2 Clayton Copula

Def: $C(u,v) = \left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-\frac{1}{\alpha}}$ for $\alpha > 0$

Generator function: $\psi(t) = \frac{1}{\alpha}\left(t^{-\alpha} - 1\right)$ where $\alpha \geq -1$ and $\alpha \neq 0$

- Lower tail dependence $\Leftarrow \alpha$

If $\alpha > 0$, $\lambda_L = 2^{-\frac{1}{\alpha}}$, lower tail dependence

$-1 \leq \alpha < 0$, $\lambda_L = 0$, no lower tail dependence

## 2.3 Frank Copula

Def: $$C(u,v) = -\frac{1}{\alpha}\ln\left[1 + \frac{(e^{-\alpha u}-1)(e^{-\alpha v}-1)}{(e^{-\alpha}-1)}\right] \quad \text{for } \alpha \neq 0$$

Generator function: $$\varphi(t) = -\ln\left(\frac{e^{-\alpha t}-1}{e^{-\alpha}-1}\right) \quad \alpha \neq 0$$

- no upper or lower tail dependence

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## 3. Implicit Copulas

No simple closed form expressions

- The Gaussian copula – based on multivariate normal distribution
- The Student t copula – based on multivariate Student's t distribution

### 3.1 The Gaussian Copula

Def: $$C(u,v) = \Phi_\rho\left[\Phi^{-1}(u), \Phi^{-1}(v)\right] = C_{Gauss}(u,v)$$

$\Phi$ is the distribution function of standard normal CDF

$\Phi_\rho$ .... bivariate Normal corr $= \rho$

$\rho = 0, 1, -1$

Q: Slide 30

$C_{Gauss}[u, v]$ , $\rho = 0 \Rightarrow$ Independent Copula $\Rightarrow$ $C_{Gauss}[u, v] = \cancel{u,v}\ uv$

i. $C_{Gauss}(1, 1) = 1 \times 1 = 1$

ii. $C_{Gauss}(1, 0.2) = 1 \times 0.2 = 0.2$

iii. $C_{Gauss}(0.2, 0.2) = 0.2 \times 0.2 = 0.04$

$\rho = 1 \Rightarrow$ Comonotonic Copula $\Rightarrow C_{Gauss}(u, v) = \min(u, v)$

$C_{Gauss}(1, 1) = \min(1, 1) = 1$

$C_{Gauss}(1, 0.2) = \min(1, 0.2) = 0.2$

## 3.2 Student's t Copula

Def: $C(u, v) = t_{\gamma,\rho}\left[t_{\gamma}^{-1}(u), t_{\gamma}^{-1}(v)\right]$

$t_\gamma$ Student's t distribution with $\gamma$ degree of freedom

$t_{\gamma,\rho}$ $\rho$: correlation bivariate Student's t distribution

- Allows the dependencies adjusted more finely
  additional parameter $\gamma$
- Gaussian Copula is the limiting case of student's t copula
  $\because$ Normal distribution is the limiting case of Student's t distribution