Queen Mary University of London

MTH5114 **Linear Programming and Games, Spring 2024**

Week 5 Seminar Questions **Viresh Patel**

Practice Questions: Solve the following linear programs using the simplex algorithm. You should give the initial tableau and each further tableau produced during the execution of the algorithm. If the program has an optimal solution, give this solution and state its objective value. If it does not have an optimal solution, say why.

You should indicate the highlighted row and columns in each pivot step as well as the row operations you carry out. This is in order to gain credit (e.g. in an exam) even if the final answer is incorrect.

1.
$$\begin{aligned} \text{maximize} \quad & 5x_1 + 6x_2 + 9x_3 + 8x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_4 \leq 5, \\ & x_1 + x_2 + 2x_3 + 3x_4 \leq 3, \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solution: Here is a list of tableaux that are produced. In each one we have also shown the calculated values to the right and the highlighted rows and columns leading to the next step.

	x_1	x_2	x_3	x_4	s_1	s_2		
s_1	1	2	3	1	1	0	5	$\frac{5}{3}$
s_2	1	1	2	3	0	1	3	$\frac{3}{2}$
	5	6	9	8	0	0	0	

	x_1	x_2	x_3	x_4	s_1	s_2		
s_1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{7}{2}$	1	$-\frac{3}{2}$	$\frac{1}{2}$	1
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	3
	$\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{11}{2}$	0	$-\frac{9}{2}$	$-\frac{27}{2}$	

	x_1	x_2	x_3	x_4	s_1	s_2		
x_2	-1	1	0	-7	2	-3	1	
x_3	1	0	1	5	-1	2	1	$\frac{1}{5}$
	2	0	0	5	-3	0	-15	

	x_1	x_2	x_3	x_4	s_1	s_2		
x_2	$\frac{2}{5}$	1	$\frac{7}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$	6
x_4	$\frac{1}{5}$	0	$\frac{1}{5}$	1	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	1
	1	0	-1	0	-2	-2	-16	

	x_1	x_2	x_3	x_4	s_1	s_2	
x_2	0	1	1	-2	1	-1	2
x_1	1	0	1	5	-1	2	1
	0	0	-2	-5	-1	-4	-17

The optimal solution is thus $x_1 = 1$, $x_2 = 2$, and $x_3, x_4, s_1, s_2 = 0$ with objective value 17.

2.
$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 \le 4, \\ & 2x_1 + 3x_3 \le 5, \\ & 2x_1 + x_2 + 3x_3 \le 7, \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution: Here is a list of tableaux that are produced. In each one we have also shown the calculated values to the right and the highlighted rows and columns leading to the next step.

	x_1	x_2	x_3	s_1	s_2	s_3		
s_1	1	1	2	1	0	0	4	2
s_2	2	0	3	0	1	0	5	$\frac{5}{3}$
s_3	2	1	3	0	0	1	7	$\frac{7}{3}$
	3	2	4	0	0	0	0	

	x_1	x_2	x_3	s_1	s_2	s_3		
s_1	$-\frac{1}{3}$	1	0	1	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{2}{3}$
x_3	$\frac{2}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{5}{3}$	
s_3	0	1	0	0	-1	1	2	2
	$\frac{1}{3}$	2	0	0	$-\frac{4}{3}$	0	$-\frac{20}{3}$	

	x_1	x_2	x_3	s_1	s_2	s_3		
x_2	$-\frac{1}{3}$	1	0	1	$-\frac{2}{3}$	0	$\frac{2}{3}$	
x_3	$\frac{2}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{5}{2}$
s_3	$\frac{1}{3}$	0	0	-1	$-\frac{1}{3}$	1	$\frac{4}{3}$	4
	1	0	0	-2	0	0	-8	

	x_1	x_2	x_3	s_1	s_2	s_3	
x_2	0	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_1	1	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{5}{2}$
s_3	0	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	1	$\frac{1}{2}$
	0	0	$-\frac{3}{2}$	-2	$-\frac{1}{2}$	0	$-\frac{21}{2}$

The optimal solution is thus $x_1 = \frac{5}{2}$, $x_2 = \frac{3}{2}$, $s_3 = \frac{1}{2}$, and $x_3, s_1, s_2 = 0$, with objective value $\frac{21}{2}$.

3.
$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + -2x_2 + x_3 \leq 1, \\ & -x_1 + x_2 - 2x_3 \leq 4, \\ & -3x_1 + -3x_2 + 2x_3 \leq 4, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution: Here is a list of tableaux that are produced. In each one we have also shown the calculated values to the right and the highlighted rows and columns leading to the next step.

	x_1	x_2	x_3	s_1	s_2	s_3		
s_1	2	−2	1	1	0	0	1	1
s_2	−1	1	−2	0	1	0	4	
s_3	−3	−3	2	0	0	1	4	2
$-z$	2	1	3	0	0	0	0	

	x_1	x_2	x_3	s_1	s_2	s_3		
x_3	2	−2	1	1	0	0	1	
s_2	3	−3	0	2	1	0	6	
s_3	−7	1	0	−2	0	1	2	2
$-z$	−4	7	0	−3	0	0	−3	

	x_1	x_2	x_3	s_1	s_2	s_3	
x_3	−12	0	1	−3	0	2	5
s_2	−18	0	0	−4	1	3	12
x_2	−7	1	0	−2	0	1	2
$-z$	45	0	0	11	0	−7	−17

Now, we cannot continue, because there are no bounds to select the leaving variable. Thus, the linear program is *unbounded*.