MTH6127 **Metric Spaces and Topology**

Course work 6 **1 March 2024**

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1. Suppose that (X, d) is a metric space and $A \subseteq X$. Show that if $A \subseteq F \subseteq X$ where F is closed, then $\bar{A} \subseteq F$, where $\bar{A}$ is the closure of A.

   $\overline{A}$ is defined as the intersection of all closed subsets containing A. Thus $\overline{A} \subset F$.

2. Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine $\bar{A}$. Is A closed?

   The closure $\overline{A}$ equals $\{0\} \cup A$. The set A is not closed as it is not equal to its closure.

3. Let (X, d) be a metric space and $A \subset X$. Then the distance from x to A is defined as

   $$\mathrm{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

   Prove that $\overline{A} = \{x \in X; \mathrm{dist}(x, A) = 0\}$.

   Suppose that $x \in \overline{A}$. Then there exists a sequence $a_n \in A$ converging to x. Then $\lim d(a_n, x) = 0$ implying that $\inf\{d(x, a); a \in A) = 0$.

   Conversely, if $\inf\{d(x, a); a \in A) = 0$ then for some sequence $a_n \in A$ we have $d(a_n, x) \to 0$. Then $a_n \to x$, i.e. $x \in \overline{A}$.

4. The diameter of a metric space (X, d) is defined as $\sup\{d(x, y); x, y \in X\}$.

   A set A in a metric space (X, d) is called bounded iff $\mathrm{diam}(A) < \infty$. Prove that:

   (a) A is bounded if and only if there exist $x \in A$ and $r > 0$ such that $A \subset B(x; r)$,
   If $D = \mathrm{diam}(A) < \infty$ then $A \subset B(x; D)$ for any $x \in A$. Conversely, if for some $x \in A$ and $r > 0$ one has $A \subset B(x; r)$ then $\mathrm{diam}(A) \le \mathrm{diam}(B(x; r)) = 2r$.

   (b) Any finite set A is bounded,
   This follows obviously from the definition.

(c) A Cauchy sequence in (X, d) is a bounded set.

Let (x_n) be a Cauchy sequence in X. We may find N such that $d(x_n, x_m) < 1$ for all $n, m > N$. Pick $n_0 > N$. The ball $B(x_{n_0}; 1)$ contains all points x_n except finitely many. Therefore the supremum

$$R = \sup\{d(x_{n_0}, x_n); n = 1, 2, 3, \dots\}$$

is finite, $R < \infty$. Thus $x_n \in B(x_{n_0}; R)$ for all $n = 1, 2, \dots$. The result now follows from part (a).

5. A Cauchy sequence (x_n) in a metric space (X, d) contains a subsequence (x_{n_i}) which converges to a point $z \in X$. Show that the whole sequence x_n converges to z as well.

Since (x_n) is a Cauchy sequence, the sequence $\sup\{d(x_{n_i}, x_n); n \geq n_i\} = \epsilon_i$ tends to 0, i.e. $\epsilon_i \to 0$ as $i \to \infty$. Thus, for $n \geq n_i$ we have

$$d(x_n, z) \leq d(x_n, x_{n_i}) + d(x_{n_i}, z) \leq \epsilon_i + d(x_{n_i}, z)$$

and when $i \to \infty$ one has $\epsilon_i \to 0$ and $d(x_{n_i}, z) \to 0$ implying our statement.