Linear Algebra Recap

Rank:

$M \in \mathbb{R}^{m \times n}$ – matrix.

- $\mathrm{rank}(M) = \dim(\text{column-space})$
  $= \dim(\text{row-space})$

- $\mathrm{rank}(M) = \mathrm{rank}(M^T) \leq \min(m, n)$

- M is called full-rank if
  $\mathrm{rank}(M) = \min(m, n)$

- $\mathrm{rank}(M^T M) = \mathrm{rank}(M M^T) = \mathrm{rank}(M)$

- If $\mathrm{rank}(M) = 1$ then:

$$M = \underline{u} \cdot \underline{v}^T$$

$$\underline{u} \in \mathbb{R}^m, \quad \underline{v} \in \mathbb{R}^n$$

Eigenvalues & Diagonalisation

$M \in \mathbb{R}^{n \times n}$ – square matrix.

- M is digonalisable if

$$M = P \cdot \underbrace{\Lambda}_{\text{diagonal}} \cdot P^{-1}$$

- Eigenvalues / eigenvectors:

  If $M \cdot \underline{v} = \lambda \underline{v}$ $\quad \underline{v} \neq \underline{0}$

then λ = eig-val $\underline{v}$ = eig-vec

⊛ λ-eigenspace = all λ-eig-vectors.
(vector space)

- If M has n independent eig-vecs $\underline{v}_1, \ldots, \underline{v}_n$ with eig-vals $\lambda_1, \ldots, \lambda_n$ then M is diagonalisable

and $M = P \cdot \Lambda \cdot P^{-1}$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

$$P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

- If M is symmetric then:

(1) M is diagonalisable

(2) all eig-vals are real

(3) If $\underline{v}_1, \underline{v}_2$ are eig-vecs for $\lambda_1 \neq \lambda_2$ then $\underline{v}_1 \perp \underline{v}_2$
$(\underline{v}_1^T \underline{v}_2 = 0)$

(4) M is diagonalisable by an orthogonal matrix

$$P^T P = I. \quad (P^{-1} = P^T)$$

$$\Rightarrow \quad M = P \Lambda P^T$$

$$\Rightarrow \quad M = \sum_{i=1}^{n} \lambda_i \underline{v}_i \cdot \underline{v}_i^T$$

✱ How to find eig-vals?

<u>Ex</u> $M = \begin{pmatrix} 2 & 1 \\ +3 & 0 \end{pmatrix}$

Note! $M\underline{v} = \lambda \underline{v} \iff (M - \lambda I)\underline{v} = \underline{0}$

$\iff \det(M - \lambda I) = 0$

$$\det(M - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ +3 & -\lambda \end{pmatrix}$$

$$= -\lambda(2-\lambda) - 3 = \lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\Rightarrow \lambda_1 = 3 \quad \lambda_2 = -1$$

Find eig-vecs: $\underline{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}$

$\lambda_1 = 3$: $(M - 3I)\underline{V_1} = \underline{0}$

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underline{0}$$

$$\Rightarrow \underline{V_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1$: $(M + I)\underline{V_2} = \underline{0} \Rightarrow \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underline{0}$

$$\underline{V_2} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

$$\Rightarrow M = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

SINGULAR VALUE DECOMPOSITION

"digonalisation" for non-square matrices.

$M \in \mathbb{R}^{m\times n} \Rightarrow M = U \cdot \Sigma \cdot V^T$

where: $U \in \mathbb{R}^{m\times m}$ $V \in \mathbb{R}^{n\times n}$ - orthogonal

$\Sigma =$ "digonal" $\in \mathbb{R}^{m\times n}$

$m \leq n$: $\Sigma = \left(\begin{array}{cc:c} \sigma_1 & 0 & \\ & \ddots & 0 \\ 0 & \sigma_m & \end{array}\right)$

$m \geq n$: $\Sigma = \left(\begin{array}{cc} \sigma_1 & 0 \\ & \ddots \\ 0 & \sigma_n \\ \hdashline & 0 \end{array}\right)$

where: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$

$r = \min(n, m)$ → singular values of M.

FACT:

$\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2$ $r = \min(m, n)$

are the eigenvalues of

both $M \cdot M^T$ and $M^T M$

- The columns of $\overset{U}{V}$ are eig-vecs of MM^T
- The colums of V are eig-vecs of M^TM.