Lemma

If $U \sim \text{Uniform}[0,1]$

then $F^{-1}(U)$ is a random variable with c.d.f. F

proof

$$P(F^{-1}(U) \leq y) \overset{\leq}{=} P(U \leq F(y))$$

because F is monotone increasing.

Now $0 \leq F(y) \leq 1$ because F is a c.d.f.

We know $P(U \leq x) = x$ for $0 \leq x \leq 1$,

so $P(F^{-1}(U) \leq y) = F(y)$. $\blacksquare$

Example

$X \sim Exp(\lambda)$, then $F(x) = 1 - e^{-\lambda x}$

Set $y = 1 - e^{-\lambda x}$

$\Rightarrow$ $e^{-\lambda x} = 1 - y$

$x = -\frac{1}{\lambda} \log(1-y)$

$\Rightarrow F^{-1}(y) = -\frac{1}{\lambda} \log(1-y)$

$F^{-1}(U) = -\frac{1}{\lambda} \ln(1-U) \sim Exp(\lambda)$

For Loops in R

For loops are a way of repeating the same operation over and over again. the structure of a for loop is

for (<variable> in <vector> {do command})

<vector> is a set of values for which we want the command repeated.

(variable) takes values in <vector>.

Bootstrapping

Suppose we want to make inferences about $\phi(\theta_1, .., \theta_p)$ from observed data. We have been assuming that the Y_i's are taken from some distribution and using the data $y_1, .., y_n$ to estimate the $\theta_1, .., \theta_p$.

An alternative method is called bootstrapping or the bootstrap method.

In this method, we form an empirical sampling distribution of ϕ from the Y_i's. We sample from the Y_i's over and over again with replacement. This is called resampling. For each resample, we estimate ϕ again.

The procedure we follow given $y_1, \dots, y_n$ is

1. The empirical distribution is
$$\hat{F}_n(y) = \frac{1}{n} \left| \{ 1 \le i \le n : y_i \le y \} \right|$$

(A) Draw a sample of size n from $\hat{F}_n(y)$. This is equivalent to sampling from $y_1, \dots, y_n$ with replacement.

(B) From this bootstrap sample form an estimate ϕ^* of ϕ in the same way we formed $\hat{\phi}$ from $y_1, \dots, y_n$

Repeat (A) and (B) B times obtaining B estimates $\phi_1^*, \dots, \phi_B^*$ of ϕ.

Now we can

1. estimate the mean of $\hat{\phi}$ by $\frac{1}{B}\sum_{i=1}^{B} \phi_i^*$

2. estimate the median of $\hat{\phi}$ by the 0.5 empirical quantile of the ϕ_i^*.

Parametric Bootstrap

We assume the X_i's come from a distribution with parameter Θ.
We estimate Θ by method of moments or maximum likelihood to obtain $\hat{\Theta}$.

Now we generate samples from the distribution with parameter $\hat{\theta}$ and proceed as before

Example

Suppose $Y_i \sim$ Exponential (λ)

our sample is $0.61, 6.47, \ldots, 0.75$

$$\bar{y} = \frac{1}{0.3530}$$

$$\hat{\lambda} = \frac{1}{\bar{y}} = 0.3530$$

We now use Exponential (0.3530) to generate new samples, and proceed as before.