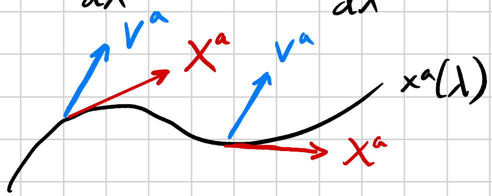
# WEEK 6

· Parallel transport and geodesics

In flat space, parallel transport of a vector $V^a$ along a curve $x^a(\lambda)$ means to "keep the vector constant along the curve".

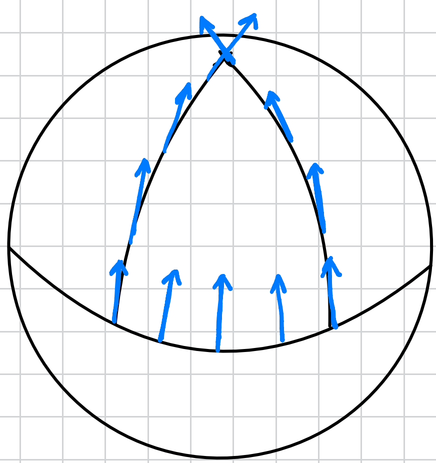
→ tangent vector to curve $x^a(\lambda)$: $X^a = \frac{dx^a}{d\lambda}$

$$\Rightarrow \quad \frac{d}{d\lambda} V^a = \frac{dx^b}{d\lambda} \partial_b V^a = X^b \partial_b V^a = 0$$




· Two issues with the above in curved spaces:

- The parallel transport depends on the path taken



- The expression above for the parallel transport is not covariant → it is NOT tensorial

· <u>Def</u>: In a general curved smooth manifold, a vector $V^a$ is said to be parallely transported along another vector $W^a$ if

$$W^b \nabla_b V^a = 0$$

· This concept can be generalised to tensors of arbitrary rank: a $(k,l)$ tensor $T$ is said to be parallely transported along a vector $W^a$ if

$$W^b \nabla_b T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = 0$$

## - Geodesics

· Straight lines in flat space are characterised by the fact that their tangent vector is parallely transported at every point:

· <u>Def</u>: Affine geodesics: curves along which the tangent vector is propagated parallely to itself:

$$W^b \nabla_b W^a = 0 \quad , \quad W^a : \text{tangent vector}$$

Explicitly: $W^a = \dfrac{dx^a}{d\lambda}$

$$\begin{aligned} W^b \nabla_b W^a &= W^b \partial_b W^a + \Gamma^a{}_{bc} W^b W^c \\ &= \frac{d}{d\lambda}\left(\frac{dx^a}{d\lambda}\right) + \Gamma^a{}_{bc} \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} \\ &= \frac{d^2 x^a}{d\lambda^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0 \end{aligned}$$

where we have used that $\dfrac{d}{d\lambda} = \dfrac{dx^a}{d\lambda}\partial_a$

→ that's the famous geodesic equation

· From the existence and uniqueness of solutions to ODEs, to every direction at a point there exists a unique geodesic passing through that point. The initial conditions are

$$\lambda = 0 \ , \quad x^a(0) = x_0^a \ , \quad \left.\frac{dx^a}{d\lambda}\right|_{\lambda=0} = W_0^a$$

• <u>Example</u> : reparametrisation of a geodesic : $\lambda \to \sigma(\lambda)$

→ The geodesic equation above only keeps the same form iff $\sigma = a\lambda + b$ , $a, b \in \mathbb{R}$

To see this, consider

$$\frac{dx^a}{d\lambda} = \frac{dx^a}{d\sigma}\frac{d\sigma}{d\lambda}$$

$$\frac{d^2x^a}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dx^a}{d\sigma}\frac{d\sigma}{d\lambda}\right) = \frac{d^2x^a}{d\lambda^2}\left(\frac{d\sigma}{d\lambda}\right)^2 + \frac{dx^a}{d\sigma}\frac{d^2\sigma}{d\lambda^2}$$

Substituting in the geodesic eq. gives

$$\left(\frac{d^2x^a}{d\sigma^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma}\right)\left(\frac{d\sigma}{d\lambda}\right)^2 + \frac{dx^a}{d\sigma}\frac{d^2\sigma}{d\lambda^2} = 0$$

This has the same form as before iff

$$\frac{d^2\sigma}{d\lambda^2} = 0 \quad \Rightarrow \quad \sigma = a\lambda + b$$

• a parameter $\lambda$ for which the geodesic equation has the form

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0$$

is called affine parameter and the geodesic is said to be affinely parametrised.

The non-affinely parametrised geodesic equation is

$$W^b \nabla_b W^a = f(\lambda) W^a$$

$$\Longleftrightarrow \quad \frac{d^2x^a}{d\lambda^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = f(\lambda) \frac{dx^a}{d\lambda}$$

$f(\lambda)$ : arbitrary function on the curve

**Note**: it is always possible to reparametrise a geodesic so that it is affinely parametrised.

· Metric geodesics

→ We will show that on a manifold M with a metric, geodesics extremise the distance between two points.

Consider the functional (i.e., function of functions)

$$\mathcal{L} = \int_{x_1}^{x_2} d\lambda \, L(x, \dot{x}, \lambda) \qquad \text{where } x = x(\lambda), \ \dot{x} = \frac{dx}{d\lambda}$$

We want to consider variations of the curve $x = x(\lambda)$ keeping endpoints fixed:

assume that $\delta$ and $\frac{d}{d\lambda}$ commute

$$\delta\mathcal{L} = \int_{x_1}^{x_2} d\lambda \, \delta L(x, \dot{x}, \lambda) = \int_{x_1}^{x_2} d\lambda \left( \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}} \delta\dot{x} \right) =$$

$$= \int_{x_1}^{x_2} d\lambda \left( \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}} \frac{d}{d\lambda}\delta x \right) = \left| \text{integrate by parts} \right| =$$

$$= \int_{x_1}^{x_2} d\lambda \left( \frac{\partial L}{\partial x}\delta x + \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x \right)$$

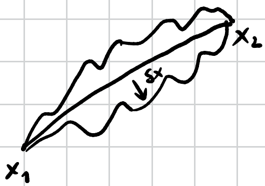
$$= \int_{x_1}^{x_2} d\lambda \left[ \frac{\partial L}{\partial x} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \delta x + \frac{\partial L}{\partial \dot{x}} \cancel{\delta x} \Big|_{x_1}^{x_2}$$

0 because $\delta x = 0$ at the endpoints

$\Rightarrow$ $\mathcal{L}$ is stationary, i.e., $\delta \mathcal{L} = 0$, for arbitrary variations $\delta x$ of the curve $x(\lambda)$ if

$$\boxed{\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0}$$

Euler-Lagrange eqs for $L$




Length of a curve: $\mathcal{L} = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} d\lambda \, \frac{ds}{d\lambda}$

but $ds^2 = g_{ab}\, dx^a\, dx^b \Rightarrow ds = \sqrt{g_{ab}\, dx^a\, dx^b}$

for a curve: $x^a = x^a(\lambda)$

$$\Rightarrow \frac{ds}{d\lambda} = \sqrt{g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}} = \sqrt{g_{ab}(x)\, \dot{x}^a \dot{x}^b}$$

$$\Rightarrow \mathcal{L} = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{g_{ab}(x(\lambda))\, \dot{x}^a \dot{x}^b}$$

Same type of functional as before with

$$L = \sqrt{g_{ab}(x(\lambda))\, \dot{x}^a \dot{x}^b}$$

→ Want to find the extrema of $\mathcal{L}$

→ The extrema of $\mathcal{L}$ are the same as the extrema of $\mathcal{L}^2$

$$\mathcal{L}^2 = \int_{\lambda_1}^{\lambda_2} d\lambda \; g_{ab}(x(\lambda))\, \dot{x}^a \dot{x}^b$$

→ $L = g_{ab}(x(\lambda))\, \dot{x}^a \dot{x}^b$

E-L: $\displaystyle \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0$

$$\frac{\partial L}{\partial \dot{x}^a} = 2\, g_{ab}\, \dot{x}^b$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 2\, g_{ab}\, \ddot{x}^b + 2\,(\partial_c g_{ab})\, \dot{x}^c \dot{x}^b$$

$$\frac{\partial L}{\partial x^a} = (\partial_a g_{bc})\, \dot{x}^b \dot{x}^c$$

E-L: $2\, g_{ab}\, \ddot{x}^b + 2\,(\partial_c g_{ab})\, \dot{x}^c \dot{x}^b - (\partial_a g_{bc})\, \dot{x}^b \dot{x}^c = 0$

$$\frac{1}{2} g^{ad}\left(2\, g_{ab}\, \ddot{x}^b + 2\,(\partial_c g_{ab})\, \dot{x}^c \dot{x}^b - (\partial_a g_{bc})\, \dot{x}^b \dot{x}^c\right) = 0$$

$$\Rightarrow \quad \ddot{x}^d + \underbrace{\frac{1}{2} g^{ad}\left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}\right)}_{\Gamma^d{}_{bc}} \dot{x}^b \dot{x}^c = 0$$

$$\Rightarrow \quad \ddot{x}^d + \Gamma^d{}_{bc}\, \dot{x}^b \dot{x}^c = 0$$

Rmk: in flat space and in Cartesian coordinates $\Gamma^a{}_{bc} = 0$ so the geodesic equation becomes

$$\ddot{x}^a = 0 \rightarrow x^a = a\lambda + b \text{ : straight line}$$

Rmk: For $g_{ab}$ Riemannian, geodesics minimise the length while for $g_{ab}$ Lorentzian geodesics maximise the length.

- Example: geodesics on the unit $S^2$

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$$L = \left(\frac{ds}{d\lambda}\right)^2 = g_{ab}\, \dot{x}^a \dot{x}^b = \dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2$$

E-L: $\dfrac{d}{d\lambda}\left(\dfrac{\partial L}{\partial \dot{x}^a}\right) - \dfrac{\partial L}{\partial x^a} = 0$ with $x^a = (\theta, \phi)$

$\underline{a = \theta}$: $\dfrac{\partial L}{\partial \theta} = 2 \sin\theta \cos\theta \, \dot{\phi}^2$

$$\frac{\partial L}{\partial \dot\theta} = 2\dot\theta \quad , \quad \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot\theta}\right) = 2\ddot\theta$$

$$\Rightarrow \quad 2\ddot\theta - 2\sin\theta\cos\theta\,\dot\phi^2 = 0$$

Divide by 2: $\ddot\theta - \sin\theta\cos\theta\,\dot\phi^2 = 0$

→ That's the $a=\theta$ component of the geodesic eq:

$$\ddot x^\theta + \Gamma^\theta{}_{bc}\,\dot x^b \dot x^c = 0$$

$$\Rightarrow \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \quad , \quad \Gamma^\theta{}_{\theta\theta} = \Gamma^\theta{}_{\theta\phi} = \Gamma^\theta{}_{\phi\theta} = 0$$

$\underline{a=\phi}$: $\quad \dfrac{\partial L}{\partial \phi} = 0$

$$\frac{\partial L}{\partial \dot\phi} = 2\sin^2\theta\,\dot\phi \quad , \quad \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot\phi}\right) = 2[\sin^2\theta\,\ddot\phi + 2\sin\theta\cos\theta\,\dot\theta\dot\phi]$$

E-L: $\dfrac{d}{d\lambda}\left(\dfrac{\partial L}{\partial \dot\phi}\right) - \dfrac{\partial L}{\partial \phi} = 2\sin^2\theta\left[\ddot\phi + 2\cot\theta\,\dot\theta\,\dot\phi\right] = 0$

$$\Rightarrow \quad \ddot\phi + 2\cot\theta\,\dot\theta\,\dot\phi = 0 \Leftrightarrow \ddot x^\phi + \Gamma^\phi{}_{bc}\,\dot x^b \dot x^c = 0$$

$$\Rightarrow \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta \quad , \quad \Gamma^\phi{}_{\theta\theta} = \Gamma^\phi{}_{\phi\phi} = 0$$

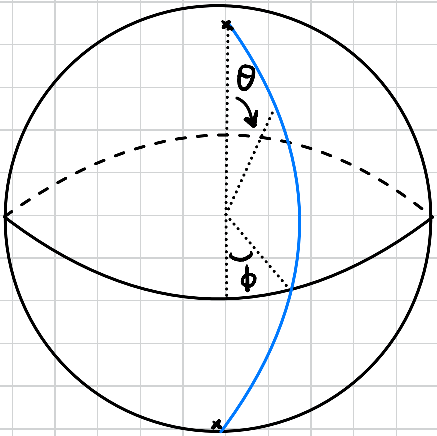
Note that because $\dfrac{\partial L}{\partial \phi} = 0$ the $a=\phi$ component of the E-L equation (and hence the geodesic equation) reduces to

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{d}{d\lambda}\left(2\sin^2\theta\,\dot\phi\right) = 0 \Rightarrow \sin^2\theta\,\dot\phi = \text{const}$$

A solution to this equation is $\dot{\phi} = 0 \Rightarrow \phi = \text{const}$

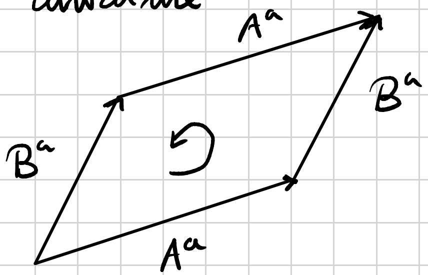
Then the $a = \theta$ component of the geodesic equation reduces to $\ddot{\theta} = 0 \Rightarrow \theta = \lambda$ by choosing the integration constants appropriately.

→ That's a meridian.



· Curvature

· The change of a vector after parallel transport around a closed loop is measured by the curvature

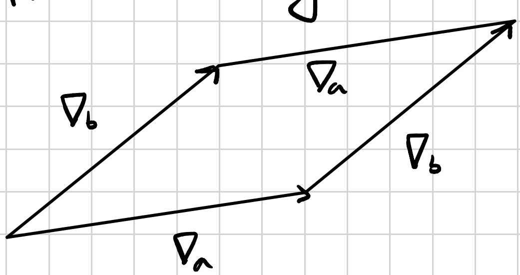



$$\delta V^c = R^c{}_{dab} V^d A^a B^b$$

$R^c{}_{dab}$ : Riemann tensor

$R^c{}_{dab} = - R^c{}_{dba}$ since interchanging the vectors $A^a$ and $B^a$ corresponds to traversing the loop in the opposite direction.

This definition is equivalent to considering the difference between parallel transport a given tensor in one direction and then the other, versus the opposite ordering:



This is measured by the commutator of two covariant derivatives. Consider an arbitrary vector field $V^a$:

$$[\nabla_a, \nabla_b] V^c = R^c{}_{dab} V^c$$

We can expand the $\nabla_a$ on the LHS to find a formula for the Riemann tensor:

$$[\nabla_a, \nabla_b] V^c = \nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c$$

$$= \partial_a \underbrace{(\nabla_b V^c)}_{\partial_b V^c + \Gamma^c{}_{bd} V^d} - \Gamma^d{}_{ab} \nabla_d V^c + \Gamma^c{}_{ad} \nabla_b V^c \quad - (a \leftrightarrow b)$$

$$= \partial_a \partial_b V^c + (\partial_a \Gamma^c{}_{bd}) V^d + \Gamma^c{}_{bd} \partial_a V^d$$
$$- \Gamma^d{}_{ab} \partial_d V^c - \Gamma^d{}_{ab} \Gamma^c{}_{de} V^e$$
$$+ \Gamma^c{}_{ad} \partial_b V^c + \Gamma^c{}_{ad} \Gamma^d{}_{be} V^e$$
$$- (a \leftrightarrow b)$$

$$= (\partial_a \Gamma^c{}_{bd} - \partial_b \Gamma^c{}_{ad} + \Gamma^c{}_{ae} \Gamma^e{}_{bd} - \Gamma^c{}_{be} \Gamma^e{}_{ad}) V^d$$

Since $V^a$ is arbitrary, we find

$$R^c{}_{dab} = \partial_a \Gamma^c{}_{bd} - \partial_b \Gamma^c{}_{ad} + \Gamma^c{}_{ae} \Gamma^e{}_{bd} - \Gamma^c{}_{be} \Gamma^e{}_{ad}$$

Note that for a scalar $[\nabla_a, \nabla_b]\phi = 0$

$$\Rightarrow \quad 0 = [\nabla_a, \nabla_b](W_c V^c) =$$

$$= \nabla_a \nabla_b (W_c V^c) - (a \leftrightarrow b)$$

$$= \nabla_a (V^c \nabla_b W_c + W_c \nabla_b V^c) - (a \leftrightarrow b)$$

$$= V^c \nabla_a \nabla_b W_c + W_c \nabla_a \nabla_b V^c$$

$$+ (\nabla_a V^c) \nabla_b W_c + (\nabla_a W_c) \nabla_b V^c - (a \leftrightarrow b)$$

$$= V^c [\nabla_a, \nabla_b] W_c + W_c [\nabla_a, \nabla_b] V^c$$

$$= V^c [\nabla_a, \nabla_b] W_c + R^c{}_{dab} V^d W_c$$

$$\Rightarrow \quad [\nabla_a, \nabla_b] W_c = - R^d{}_{cab} W_d$$

Proceeding by induction we can compute

$$[\nabla_c, \nabla_d] X^{a_1 \dots a_k}{}_{b_1 \dots b_l} = R^{a_1}{}_{ecd} X^{e a_2 \dots a_k}{}_{b_1 \dots b_l} + \dots$$

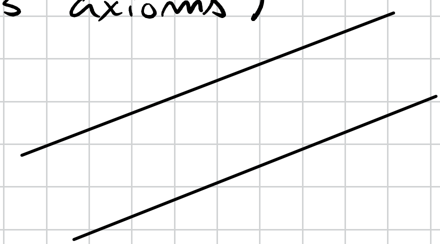
$$+ R^{a_k}{}_{ecd} X^{a_1 \dots a_{k-1} e}{}_{b_1 \dots b_l}$$

$$- R^e{}_{b_1 cd} X^{a_1 \dots a_k}{}_{e b_2 \dots b_l}$$

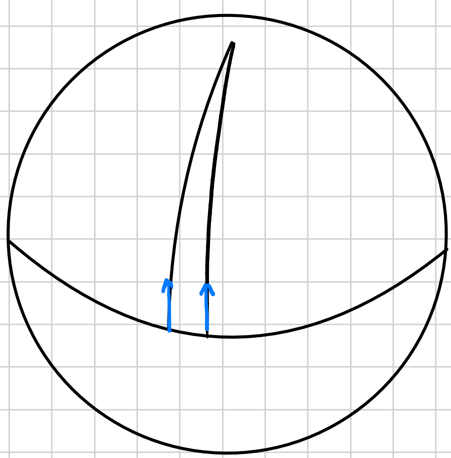
$$- R^e{}_{b_l cd} X^{a_1 \dots a_k}{}_{b_1 \dots b_{l-1} e}$$

## Geodesic deviation

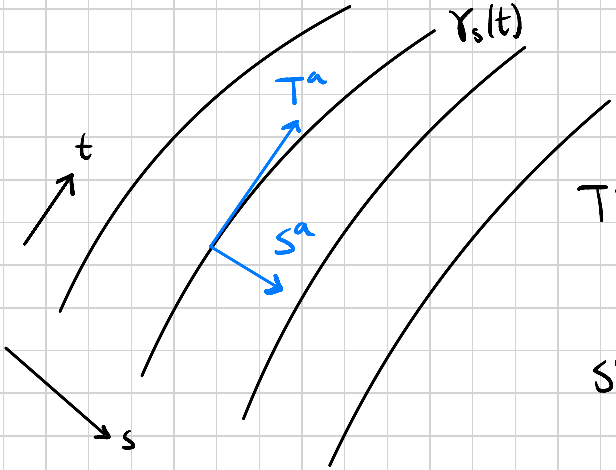
- In flat Euclidean space two straight lines that are initially parallel remain parallel (one of Euclid's axioms)



- That's NOT true in curved spaces. Curvature (i.e., Riemann tensor) measures the failure of two geodesics that are initially parallel to remain parallel



Consider a one parameter family of geodesics $\gamma_s(t)$ so that for each $s \in \mathbb{R}$, $\gamma_s$ is a geodesic with affine parameter $t$. The parameters $(s,t)$ can be chosen as coordinates on this surface:




$T^a = \dfrac{\partial x^a}{\partial t}$ : tangent vector to the geodesics

$S^a = \dfrac{\partial x^a}{\partial s}$ : deviation vector

Define the "relative velocity" of the geodesics:

$$V^a = (\nabla_T S)^a = T^b \nabla_b S^a$$

Relative acceleration of the geodesics:

$$A^a = (\nabla_T V)^a = T^b \nabla_b V^a$$

Since $S$ and $T$ are basis vectors adapted to a coordinate system, their commutator vanishes:

$$[S,T] = 0 \Rightarrow S^b \nabla_b T^a = T^b \nabla_b S^a$$

$\left( 0 = [S,T]^a = S^b \partial_b T^a - T^b \partial_b S^a \text{ w/ } T^a = \partial_t^a, S^a = \partial_s^a \right)$

We compute:

$$
\begin{aligned}
A^a &= T^b \nabla_b V^a = T^b \nabla_b (T^c \nabla_c S^a) \\
&= T^b \nabla_b (S^c \nabla_c T^a) \\
&= (T^b \nabla_b S^c) \nabla_c T^a + T^b S^c \nabla_b \nabla_c T^a \\
&= (T^b \nabla_b S^c) \nabla_c T^a + T^b S^c ( \nabla_c \nabla_b T^a + R^a{}_{dbc} T^d) \\
&= \cancel{(T^b \nabla_b S^c) \nabla_c T^a} + S^c \nabla_c \underbrace{(\cancelto{0}{T^b \nabla_b T^a})} - \cancel{(S^c \nabla_c T^b)} \nabla_b T^a \\
&\quad + R^a{}_{dbc} T^d T^b S^c
\end{aligned}
$$

$T^a$ is the tangent vector to an affinely parametrised geod.

$$= R^a{}_{dbc} T^d T^b S^c$$

→ Physically the acceleration of neighbouring geodesics is interpreted as a manifestation of the gravitational tidal forces.

## Symmetries of the Riemann tensor

Consider

$$R_{abcd} = g_{ae} R^e{}_{bcd} =$$
$$= g_{ae}\left(\partial_c \Gamma^e{}_{bd} - \partial_d \Gamma^e{}_{bc}\right) + \Gamma_{aec}\Gamma^e{}_{bd} - \Gamma_{aed}\Gamma^e{}_{bc}$$

where $\Gamma_{abd} = g_{af}\Gamma^f{}_{bd} = \frac{1}{2}\left(\partial_b g_{da} + \partial_d g_{ba} - \partial_a g_{bd}\right)$

Since $R_{abcd}$ is a tensor, it has the same symmetries in all coordinate frames. Consider a locally inertial frame, i.e., $g_{\hat{a}\hat{b}} = \mathrm{diag}(-1, 1 \dots 1)$ and $\partial_{\hat{c}} g_{\hat{a}\hat{b}}\big|_p = 0 \Rightarrow \Gamma^{\hat{a}}{}_{\hat{b}\hat{c}} = 0$,

$$\Rightarrow R_{\hat{a}\hat{b}\hat{c}\hat{d}} = g_{\hat{a}\hat{f}}\left(\partial_{\hat{c}} \Gamma^{\hat{f}}{}_{\hat{b}\hat{d}} - \partial_{\hat{d}} \Gamma^{\hat{f}}{}_{\hat{b}\hat{c}}\right)$$
$$= \frac{1}{2}\left(\partial_{\hat{b}}\partial_{\hat{c}} g_{\hat{a}\hat{d}} + \partial_{\hat{a}}\partial_{\hat{d}} g_{\hat{b}\hat{c}} - \partial_{\hat{a}}\partial_{\hat{c}} g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}} g_{\hat{a}\hat{c}}\right)$$

We can now easily read off the symmetries:

$R_{abcd} = -R_{bacd}$, $R_{abcd} = -R_{abdc}$, $R_{abcd} = R_{cdab}$

$R_{abcd} + R_{adbc} + R_{acdb} = R_{a[dbc]} = 0$

(1st Bianchi identity)

$\Rightarrow$ In 4d, $R_{abcd}$ has 20 independent components

· Bianchi identity

Recall that in a locally inertial frame,

$$R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \frac{1}{2}\left(\partial_{\hat{a}}\partial_{\hat{d}}\, g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}\, g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}\, g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}\, g_{\hat{a}\hat{d}}\right)$$

ally

$$\rightarrow \partial_{\hat{e}} R_{\hat{c}\hat{d}\hat{a}\hat{b}} = \frac{1}{2}\,\partial_{\hat{e}}\left(\partial_{\hat{a}}\partial_{\hat{d}}\, g_{\hat{c}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{c}}\, g_{\hat{b}\hat{d}} - \partial_{\hat{b}}\partial_{\hat{d}}\, g_{\hat{c}\hat{a}} + \partial_{\hat{b}}\partial_{\hat{c}}\, g_{\hat{a}\hat{d}}\right)$$

Now consider the sum of the cyclic permutations of the first three indices:

$$\partial_{\hat{e}} R_{\hat{c}\hat{d}\hat{a}\hat{b}} + \partial_{\hat{c}} R_{\hat{d}\hat{e}\hat{a}\hat{b}} + \partial_{\hat{d}} R_{\hat{e}\hat{c}\hat{a}\hat{b}} =$$

$$\begin{aligned}
= \frac{1}{2}\Big( & \partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{d}}\, g_{\hat{c}\hat{b}} - \partial_{\hat{e}}\partial_{\hat{a}}\partial_{\hat{c}}\, g_{\hat{b}\hat{d}} - \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{d}}\, g_{\hat{c}\hat{a}} + \partial_{\hat{e}}\partial_{\hat{b}}\partial_{\hat{c}}\, g_{\hat{a}\hat{d}} \\
& + \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{e}}\, g_{\hat{d}\hat{b}} - \partial_{\hat{c}}\partial_{\hat{a}}\partial_{\hat{d}}\, g_{\hat{b}\hat{e}} - \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{e}}\, g_{\hat{d}\hat{a}} + \partial_{\hat{c}}\partial_{\hat{b}}\partial_{\hat{d}}\, g_{\hat{a}\hat{e}} \\
& + \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{c}}\, g_{\hat{e}\hat{b}} - \partial_{\hat{d}}\partial_{\hat{a}}\partial_{\hat{e}}\, g_{\hat{b}\hat{c}} - \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{c}}\, g_{\hat{e}\hat{a}} + \partial_{\hat{d}}\partial_{\hat{b}}\partial_{\hat{e}}\, g_{\hat{a}\hat{c}}\Big)
\end{aligned}$$

$$= 0$$

This is a tensor equation and hence it should be true in any coordinate system:

$$\nabla_e R_{cdab} + \nabla_d R_{ecab} + \nabla_c R_{deab} = \nabla_{[e} R_{cd]ab} = 0$$

→ 2nd Bianchi identity

· The Ricci tensor

$$R_{ab} = g^{cd} R_{cadb} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{cb} + \Gamma^d{}_{ab} \Gamma^c{}_{cd} - \Gamma^d{}_{ca} \Gamma^c{}_{db}$$

Rmk: $R_{ab} = R_{ba}$

Note that $\Gamma^a{}_{ab} = \partial_b \ln\sqrt{|g|}$ where $g = \det g_{ab}$. Then

$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \partial_b \ln\sqrt{|g|} + \Gamma^d{}_{ab} \partial_d \ln\sqrt{|g|} - \Gamma^d{}_{ca} \Gamma^c{}_{db}$$

· The Ricci Scalar : $R = g^{ab} R_{ab} = g^{ac} g^{bd} R_{abcd}$

· The Einstein tensor : $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$

· $\nabla^a G_{ab} = 0$

· Proof : Contract twice the 2nd Bianchi identity

$$\begin{aligned} 0 &= g^{bd} g^{ae} \left( \nabla_e R_{cdab} + \nabla_c R_{deab} + \nabla_d R_{ecab} \right) \\ &= \nabla^a R_{ca} + \nabla_c R + \nabla^b R_{cb} \\ &= 2 \left( \nabla^a R_{ac} + \frac{1}{2} \nabla_c R \right) = 2 \nabla^a G_{ac} \end{aligned}$$

→ The fact that $G_{ab}$ is divergence free is a geometric property!

- ## The Weyl tensor

The Ricci tensor and the Ricci scalar contain all the information about the contractions of the Riemann tensor. The Weyl tensor is the trace-free part of the Riemann:

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\left( g_{a[c} R_{d]b} - g_{b[c} R_{d]a} \right) + \frac{2}{(n-2)(n-1)} R\, g_{a[c}\, g_{d]b}$$

- $C^{a}{}_{bac} = 0$
- The Weyl tensor has the same symmetries as the Riemann:

$$C_{abcd} = C_{[ab][cd]}\ ,\quad C_{abcd} = C_{cdab}\ ,\quad C_{a[bcd]} = 0$$

- The Weyl tensor is invariant under conformal transformations of the metric: $g_{ab} \to \Omega(x)^2 g_{ab}$