Assessed coursework 1

hand in your work by
11 am, 4th of March

The problem set can be found in
the Week 5 tab of the MTH 4104
QMplus page.

Last week, I defined:

<u>Def</u> A group $(G, *)$

is a set G

with operation $*$

satisfying

(G0) if $a, b \in G$,

$$a * b \in G$$

(G1) if $a, b, c \in G$.

$$a * (b * c) = (a * b) * c$$

(G2) There is an element e of G

s.t. $e * a = a * e = a$

$\forall\ a \in G$

(G3) For every element a in G,

there exists $b \in G$ s.t.

$$a * b = b * a = e.$$

If $(G, *)$ satisfies

(G4) $\forall\ a, b \in G$, $a * b = b * a$,

we call it an abelian group.

Prop 1.4 $*$ the identity element in (G2)

is unique.

* The inverse of a is unique

$\forall a \in G$

and two more assertions.

Rings

<u>Def</u> A ring is a set R

which comes equipped with two operations

$+$ (addition)

$\times$ (multiplication)

(R+0) If $a, b \in R$,

then $a+b \in R$.

(R+1) If $a, b, c \in R$,

then $a+(b+c) = (a+b)+c$

in R.

(R+2) There is an element 0 in R

satisfying $a+0 = 0+a = a$

$\forall a \in R$.

(R+3) For every element a in R

there exists b in R s.t.

$$a+b=b+a=0$$

(R+4) $\forall a, b \in R,$

$$a+b=b+a.$$

(Rx0) $\forall a, b \in R$

$$a \times b \in R.$$

(Rx1) If a, b, c are elements in R,

then $a \times (b \times c) = (a \times b) \times c$

$(R\times +)$ If $a, b, c \in R$, then

$$a \times (b+c) = a \times b + a \times c$$

$(R+\times)$ If $a, b, c \in R$, then

$$(b+c) \times a = b \times a + c \times a$$

Note that

$a \times (b+c)$ is not necessarily

same as $(b+c) \times a$

Rk By $(R+0) - (R+4)$,

$$(G, *) = (R, +)$$

is a group

(A ring is a group).

<u>Rk</u> I shall write ab

for $a \times b$ from now on

<u>Def</u> A ring $(R, +, \times)$

is a commutative ring

if $\forall a, b \in R$, $ab = ba$.

Examples:

0, or the identity element
w.r.t. "+",
needs to be in a ring R.

- $\{0\}$ $\quad 0+0=0$
  $0 \times 0 = 0$

  This is the smallest possible ring.

- $(\mathbb{Z}, +, \times)$ is a ring.

- The set $\mathbb{C}[x]$ of polynomials in one variable X with coeffs in $\mathbb{C}$,

$$c_n X^n + c_{n-1} X^{n-1} + \cdots + c_1 X + c_0$$

$$c_i \in \mathbb{C}$$

$$(X^2 + X + 1) + (X^3 + 3X)$$

$$= X^3 + X^2 + 4X + 1 \quad \text{etc.}$$

$$(X+1)X$$

$$= X^2 + X \quad \text{etc.}$$

- The set $M_2(\mathbb{C})$ of 2-by-2

matrices with entries in $\mathbb{R}$

is a ring:

$+$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$ $\in$

$a, b, c, d \in \mathbb{C}$

$M_2(\mathbb{R})$

$\times$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ $\in M_2(\mathbb{R})$

$= \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}$

0 : $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R})$

$\times$ to be $\forall a, b \in G$

$$a \times b = e$$

where e is the identity element

in $(G, *)$.

Then $(G, \underset{\shortparallel}{+}, \times)$ is a commutative ring.
$*$

because $ab = e$

$ba = e$

so $ab = ba = e$.

- R = the set of non-negative integers
  $0, 1, 2, \dots$

  $+$

  $\times$ as we know for integers

Is $(R, +, \times)$ a ring?

This is not a ring. It fails on (R+3)

First of all, 0 is 0 in (R+2) but 2 does NOT

- R = the set of positive integers
  $1, 2, 3, \dots$

  $+$ b s.t.

  $\times$ as we know then $2+b=0$ in R

Is this $(R, +, \times)$ a ring? /

(-2) is NOT

a non-negative

integer.

This is not a ring either

because 0 is NOT in R

& $(R+2)$ does NOT hold.

$i = \sqrt{-1}$

$$\mathbb{Z}[i] := \{ a+bi \mid a, b \in \mathbb{Z} \}$$

$+$: $(a+bi)+(a'+b'i)$

$$= \underbrace{(a+a')}_{\in \mathbb{Z}} + \underbrace{(b+b')}_{\in \mathbb{Z}} i$$

$\times$: $(a+bi)(a'+b'i)$

$$= \underbrace{(aa'-bb')}_{\in \mathbb{Z}} + \underbrace{(a'b+ab')}_{\in \mathbb{Z}} i$$

0 (as in $R\neq 2$) : $0 + 0i$

$\parallel$

0.

This is called the ring of

Gaussian integers $\square$