MTH5113 (2023/24): Problem Sheet 6

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 2**.

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(1) *(Warm-up)* For each of the sets C and points $\mathbf{p} \in C$ given below:

(i) Show that C is a curve.

(ii) Sketch C, and indicate the point $\mathbf{p}$ on C.

(iii) Give a parametrisation of C that passes through $\mathbf{p}$.

(a) *Hyperbola:*

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = -1\}, \qquad \mathbf{p} = (0, -1).$$

(b) *Cubic:*

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x - 2)^3 = y - 3\}, \qquad \mathbf{p} = (0, -5).$$

(c) *Ellipse:*

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 6x + 4y^2 - 8y = 3\}, \qquad \mathbf{p} = (-3, -1).$$

(2) *(Fun with plotting)* The following are exercises involving sketching parametric surfaces. Do make use of computer programs or webpages (see the links in the *Additional Resources* section on the *QMPlus page*) to help you with your sketches.

(a) *Sphere:* Consider the following parametric surface:

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u \sin v,\ \sin u \sin v,\ \cos v).$$

(i) Sketch the paths obtained from σ by holding v constant, with values

$$v_0 = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi.$$

(ii) Sketch the paths obtained from σ by holding u constant, with values

$$u_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

(iii) Sketch the image of σ.

(b) *Hyperboloid:* Consider the following parametric surface:

$$\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \mathbf{h}(u, v) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v).$$

(i) Sketch the paths obtained from $\mathbf{h}$ by holding v constant, with values

$$v_0 = -2, -1, 0, 1, 2.$$

(ii) Sketch the paths obtained from $\mathbf{h}$ by holding u constant, with values

$$u_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

(iii) Sketch the image of $\mathbf{h}$.

(3) *(Warm-up)* For each of the following parametric surfaces σ and parameters (u_0, v_0), compute the tangent plane to σ at (u_0, v_0):

(a) σ is the parametric *cylinder*,

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u,\ \sin u,\ v),$$

and $(u_0, v_0) = (\frac{\pi}{2}, -1)$.

(b) σ is the parametric *one-sheeted hyperboloid*,

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v).$$

and $(u_0, v_0) = (\pi, 1)$.

(4) *(Introduction to curve integrals)* One can also define an intermediate notion of curve integration of vector fields over *parametric curves*. More specifically:

Definition. Let $\gamma : (a, b) \to \mathbb{R}^n$ be a parametric curve, and let $\mathbf{F}$ be a vector field that is defined on the image of γ. We then define the *curve integral* of $\mathbf{F}$ over γ by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b [\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}]\, dt.$$

For each of the following γ and $\mathbf{F}$, compute the curve integral of $\mathbf{F}$ over γ:

(a) γ is the regular parametric curve

$$\gamma : (0, 1) \to \mathbb{R}^3, \qquad \gamma(t) = (t,\ -t,\ 2t),$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x, y, z)_{(x,y,z)}.$$

(b) γ is the regular parametric curve

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, \qquad \gamma(t) = (\cos t,\ \sin t \cos t),$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (x^2, 0)_{(x,y)}.$$

(5) [**Marked**] *(Conservative forces)* Conservative forces have a very special property that we will explore in this problem. Consider a function $U : \mathbb{R}^2 \to \mathbb{R}$ which we will call *the potential*. The *conservative force* $\mathbf{F}$ associated to the potential U is given by

$$\mathbf{F} = -\nabla U\ .$$

(a) Consider the following potential

$$U(x, y) = -\frac{1}{\sqrt{x^2 + y^2 + 1}}\ .$$

Compute the force $\mathbf{F}$ associated to U and sketch the vector field associated to $\mathbf{F}$. (You should produce something like Figure 2.18 from the *lecture notes*.)

(b) Consider two curves C_1 and C_2 with injective parametrizations $\gamma_1 : (0, 2\pi) \to \mathbb{R}^2$ and

$\gamma_2 : (0, 2\pi) \to \mathbb{R}^2$ given by

$$\gamma_1(t) = \left(0, \frac{t}{2\pi}\right) , \qquad \gamma_2(t) = \left(\sin(t), \frac{t}{2\pi}\right) ,$$

Sketch the curves C_1 and C_2.

(c) Compute the curve integrals $\int_{C_1} \mathbf{F} \cdot \mathbf{ds}$ and $\int_{C_2} \mathbf{F} \cdot \mathbf{ds}$. Do you notice anything?

(d) If you've done things correctly, you should have found $\int_{C_1} \mathbf{F} \cdot \mathbf{ds} = \int_{C_2} \mathbf{F} \cdot \mathbf{ds}$: *the answer is independent of the shape of the curve*! In fact, you may have noticed that

$$\int_{C_1} \mathbf{F} \cdot \mathbf{ds} = \int_{C_2} \mathbf{F} \cdot \mathbf{ds} = -\left(U(0,1) - U(0,0)\right) .$$

Does this remind you of anything? We will explore this further in section 5.4 of the lecture notes.

(6) [**Tutorial**] For each of the following oriented curves C and vector fields $\mathbf{F}$:

(i) Give an *injective* parametrisation γ of C such that *the image of* γ *differs from* C *by only a finite number of points. Which orientation does* γ *generate?*

(ii) Compute the (curve) integral of $\mathbf{F}$ over the curve C.

(a) C is the *anticlockwise-oriented ellipse*,

$$C = \{(x, y) \in \mathbb{R}^2 \mid 3x^2 + 2y^2 = 6\},$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (y, -x)_{(x,y)}.$$

(b) C is the *downward-oriented* (with decreasing z-value) *helical segment*,

$$C = \{(\cos t, \sin t, t) \mid t \in (0, 2\pi)\},$$

and $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (-y, x, 1)_{(x,y,z)}.$$

(7) *(Exploring curvature)* Let $\gamma : I \to \mathbb{R}^n$ be any regular parametric curve. We then define the *curvature* of γ at $t \in I$ by the formula

$$\kappa_\gamma(t) = \frac{1}{|\gamma'(t)|} \left| \left(\frac{\gamma'}{|\gamma'|} \right)'(t) \right| .$$

(This can be viewed as the "change in the direction of γ per unit length"; see the *2019* version of the *MTH5113 lecture notes* for additional discussions of curvature.)

(a) Let $\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$, and let ℓ be the parametric line,

$$\ell : \mathbb{R} \to \mathbb{R}^n, \qquad \ell(t) = \mathbf{p} + t\mathbf{v}.$$

Compute the curvature of ℓ at every $t \in \mathbb{R}$.

(b) Let $R > 0$, and let γ_R be the parametric circle of radius R:

$$\gamma_R : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma_R(t) = (R \cos t,\ R \sin t).$$

Compute the curvature of γ_R at every $t \in \mathbb{R}$.

(c) Show that *curvature is independent of parametrisation.* More specificially, show that if γ is a reparametrisation of $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$, with corresponding change of variables $\phi : I \to \tilde{I}$ (in particular, $\gamma(t) = \tilde{\gamma}(\phi(t))$ for all $t \in I$), then

$$\kappa_\gamma(t) = \kappa_{\tilde{\gamma}}(\phi(t)), \qquad t \in I.$$

(8) *(Polar curves)* Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth positive periodic function, with period 2π:

$$h(\theta) > 0, \qquad h(\theta + 2\pi) = h(\theta), \qquad x \in \mathbb{R}.$$

Let the *polar curve* P be the set of all points in $\mathbb{R}^2$ satisfying the relation

$$r = h(\theta)$$

in polar coordinates. (Here, you can assume that P is indeed a curve.)

(a) The unit circle $\mathcal{C} = \{(x, y) \in \mathbb{R} \mid x^2 + y^2 = 1\}$ is a polar curve. What is h here?

(b) Give a injective parametrisation of P whose image is all of P except for a single point.

(c) Derive a formula for the arc length of P.

(9) *(Conic sections)* Let N denote the following cone:

$$N = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2\}.$$

In addition, let $P \subseteq \mathbb{R}^3$ denote an arbitrary plane that does not pass through the origin. More specifically, P is a set of the form

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\}$$

where $a, b, c, d \in \mathbb{R}$ satisfy $(a, b, c) \neq (0, 0, 0)$ and $d \neq 0$. A set of the form $N \cap P$ (i.e. the intersection of the cone N and the plane P) is called a *conic section.*

(a) Use the theorem in Question (9) of Problem Sheet 4 to show that any conic section $N \cap P$ is indeed a curve. *(Hint: You will have to be resourceful to do this. The first step is to express* $N \cap P$ *as an appropriate level set.)*

(b) Find examples of such planes P such that the conic section $N \cap P$ is:

(i) A circle.

(ii) An ellipse.

(iii) A parabola.

(iv) A hyperbola.

Check your answers graphically on a computer!