RIEMANN INT. and UNIFORM CONT/CONV

WEEK 6

Thm 6.1.11 Let $f:[a,b]\to\mathbb{R}$ be bounded and increasing (or monotone decreasing), then f is integrable.

Proof Assume f is increasing (otherwise choose $-f$)

For $\varepsilon>0$ we need to find a partition $P=\{x_0,\dots,x_n\}$ of (a,b) such that $U(f,P)-L(f,P)<\varepsilon$ (RIC)

Let P be the equipartition: $x_i = a + i\left(\frac{b-a}{n}\right)$, $i=0,\dots,n$

Then,
$$U(f,P)-L(f,P)$$
$$=\sum_{i=1}^{n}\Big(\sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x)\Big)(x_i - x_{i-1})$$

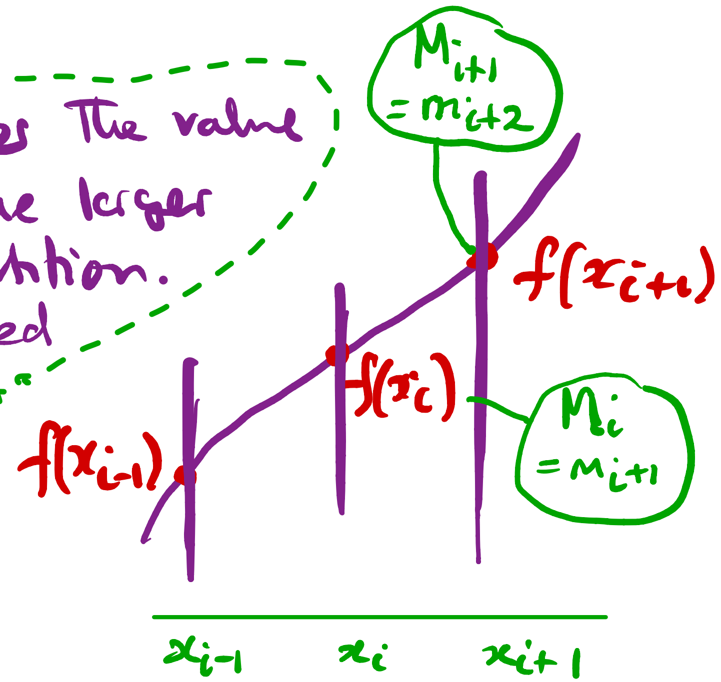
$$= \left(\frac{b-a}{n}\right)\sum_{i=1}^{n}\left(f(x_i)-f(x_{i-1})\right) \overset{*}{=} \left(\frac{b-a}{n}\right)\left(\sum_{i=1}^{n} f(x_i) - \sum_{i=0}^{n-1} f(x_i)\right)$$

$$= \left(\frac{b-a}{n}\right)\left(f(x_n)-f(x_0)\right) = \frac{(b-a)}{n}\left(f(b)-f(a)\right)$$

$$\therefore 0 \le U(f,P) - L(f,P) < \varepsilon \qquad \text{(RIC)}$$

provided $n > \dfrac{(b-a)(f(b)-f(a))}{\varepsilon}$.

The smaller the value of ε, the larger the partition needed.



$\therefore$ f is Riemann integrable

$$* \quad (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + (f(x_3) - f(x_2)) + \dots$$

$$+ \dots + f(x_{n-1}) - f(x_{n-2}) + (f(x_n) - f(x_{n-1})) = f(x_n) - f(x_0)$$

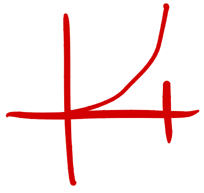
Example Consider the partition $P_n = \{ i/n \mid i = 0, \dots, n \}$

Find $U(f, P_n)$, $L(f, P_n)$ for $f(x) = x^2$ on $[0,1]$

$\boxed{f(x_i) = \frac{i^2}{n^2}}$

$\frac{1}{n} = x_i - x_{i-1}$

$$U(f,P_n) = \sum_{i=1}^{n} x_i^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2, \quad L(f,P_n) = \sum_{i=0}^{n-1} x_i^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=0}^{n-1} i^2 \quad *$$



Note, given any $\varepsilon > 0$

$$\therefore \quad U(f,P_n) - L(f,P_n) = \frac{1}{n^3}(n^2 - 0^2) = \frac{1}{n} < \varepsilon$$

provided $n > \frac{1}{\varepsilon}$ $\therefore$ By RIC (Thm 6.1.9), f is Riemann integrable, i.e. $\int_0^1 f$ exists $\left(\int_0^1 f = \int_0^1 f(x)\,dx. \right)$

$*$ Use $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$ to evaluate $U(f,P)$, $L(f,P)$

We obtain: $*$

$$L(f,P_n) = \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = U(f,P_n)$$

Taking limits as $n \to \infty$:

$$\frac{2}{6} \le \underline{\int_0^1} f \le \int_0^1 f \le \overline{\int_0^1} f \le \frac{2}{6}$$

$$? $$

$$\therefore \underline{\int_a^b} f = \overline{\int_a^b} = \frac{1}{3} \Rightarrow \int_a^b f = \frac{1}{3}$$

Note: $\int_a^b f$ exists when $\underline{\int_a^b} f = \overline{\int_a^b}$

Check using anti-derivative: $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

§6.2 Uniform Continuity

The function $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is

Def^n 6.2.1 The function $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is s.t.b. uniformly continuous on Ω if given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ provided $|x - y| < \delta$ for $x, y \in \Omega$.

Note Clearly f uniformly continuous implies $\to$ f is continuous

For f to be continuous at $x = x_0$, given ε, the choice of δ may depend on both ε and x_0.

For f to be uniformly continuous, δ will depend on ε and be independent of x_0.

Example 6.2.2

$f(x) = x$ is uniformly continuous on $\mathbb{R}$

Let $\varepsilon > 0$, we require $|f(x) - f(y)|$ $(= |x-y|)$ to be less than ε for $|x-y| < \delta$. Clearly $\delta = \varepsilon$ works!

Example 6.2.3

$f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Assume that f is uniformly continuous. i.e. given $\varepsilon > 0$ $\exists$ $\delta = \delta(\varepsilon) > 0$ such that $|x-y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$,

Suppose $x \geq 0$ choose $y = x + \delta/2$, i.e $|x-y| < \delta$.

Then $|x^2 - y^2| = |x^2 - (x^2 + \delta x + \delta^2/4)| = \left|-\left(\delta x + \frac{\delta^2}{4}\right)\right| = \delta x + \frac{\delta^2}{4} > \delta x$

If we choose $x = \varepsilon/\delta$ then $|x^2 - y^2| > \delta(\varepsilon/\delta) = \varepsilon$

i.e. $|x^2 - y^2| > \varepsilon$, contradiction!

Theorem 6.2.4 Suppose f is continuous on a closed bounded interval $\Omega = [a,b]$, then f is uniformly continuous.

Proof We want to prove

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, y \in \Omega \text{ with } |x-y| < \delta$$

implies $|f(x) - f(y)| < \varepsilon$.

We will prove by contradiction, i.e. let us assume f is not uniformly continuous

i.e. $\exists \varepsilon_0 > 0$, such that $\forall \delta$ $\exists$ $x, y \in \Omega$ with $|x-y| < \delta$ with the property that $|f(x) - f(y)| \geq \varepsilon_0$

Let $\delta = 1$, $\exists$ $x_1, y_1 \in \Omega$: $|x_1 - y_1| < 1$, but $|f(x_1) - f(y_1)| \geq \varepsilon_0$

Let $\delta = \frac{1}{2}$, $\exists$ $x_2, y_2 \in \Omega$: $|x_2 - y_2| < \frac{1}{2}$, but $|f(x_2) - f(y_2)| \geq \varepsilon_0$

Let $\delta = \frac{1}{n}$, $\exists$ $x_n, y_n \in \Omega$: $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \varepsilon_0$

and so on!

The sequence $\{x_n\}$ is bounded ($x_n \in \Omega$, $\forall n \in \mathbb{Z}^+$)

By the Bolzano-Weierstrass Thm. (not part of MTH5105) the sequence has a convergent subsequence $\{x_{n_k}\}$ with $\lim_{k\to\infty} x_{n_k} = x_0 \in \Omega$

The subsequence $\{y_{n_k}\}$ also converges to x_0

Note: $|y_{n_k} - x_0| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x_0|$

$$\underset{k\to\infty}{\downarrow 0} \quad \Leftarrow \quad \underset{k\to\infty}{\downarrow 0} \qquad \underset{k\to\infty}{\downarrow 0}$$

But f is continuous at $x = x_0$, so for sufficiently large $k >$ (some K)

$$|f(x_{n_k}) - f(x_0)| < \frac{\varepsilon_0}{4} \quad \text{and} \quad |f(y_{n_k}) - f(x_0)| < \frac{\varepsilon_0}{4}$$

It follows that for $k > K$,

$$\varepsilon_0 \leq |f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| < \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2}$$

$\varepsilon_0 \leq \frac{\varepsilon_0}{2}$ - contradiction!

Theorem 6.2.5 Every continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable $(\Omega = [a,b])$

Proof Since $[a,b]$ is closed and bounded, by Thm 6.2.4, f is uniformly continuous. This means

$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0$ such that $x, y \in \Omega$ with $|x-y| < \delta$

$\Rightarrow \quad |f(x) - f(y)| < \cancel{\varepsilon}\,/(b-a)$ mesh(P)

Now choose a partition P | $\sigma(P) = \sup_i |x_i - x_{i-1}| < \delta$

On each interval $I_i = [x_{i-1}, x_i]$, f is continuous and achieves its bounds

$$\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) = f(\bar{x}) - f(\bar{y}), \text{ some } \bar{x}, \bar{y} \in [x_{i-1}, x_i] \in I_i$$

Note $|\bar{x}-\bar{y}|<\delta \therefore |f(\bar{x})-f(\bar{y})| < \dfrac{\varepsilon}{|b-a|}$

or $M_i - m_i < \dfrac{\varepsilon}{|b-a)}$. $M_i = f(\bar{x}_i)$

$m_i = f(\bar{y}_i)$

$\therefore U(f,P) - L(f,P) < \displaystyle\sum_{i=1}^{n} \frac{\varepsilon}{(b-a)}(x_i - x_{i-1}) = \frac{\varepsilon}{(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1})$

$= \dfrac{\varepsilon}{(b-a)}(b-a) = \varepsilon.$ (RIC)

$\therefore f$ is Riemann integrable

§7 Properties of the Riemann Integral.

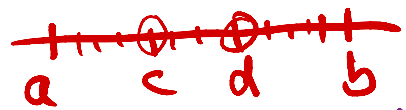
§7.1 Properties

Thm 7.1.1 Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. If $[c,d] \subseteq [a,b]$, then f is Riemann integrable on $[c,d]$.

Proof $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$

$$U(f,P) - L(f,P) < \varepsilon \qquad \text{(RIC)}$$

let $P' = P \cup \{c,d\}$, a refinement of P, then



$P' = \{x_0 = a, x_1, \dots, x_k = c, \dots, x_{k+r} = d, x_{k+r+1}, \dots, x_n\}$, and

$$U(f,P') - L(f,P') < \varepsilon$$

Now let $P'' = \{c = x_k, \dots, x_{k+r} = d\}$ a partition of $[c,d]$.

$$\therefore \quad U(f,P'') - L(f,P'') = \sum_{i=k+1}^{k+r} (M_i - m_i)(x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i+1})$$

$$= U(f,P') - L(f,P') < \varepsilon$$

$\Rightarrow f$ is integrable on $[c,d]$

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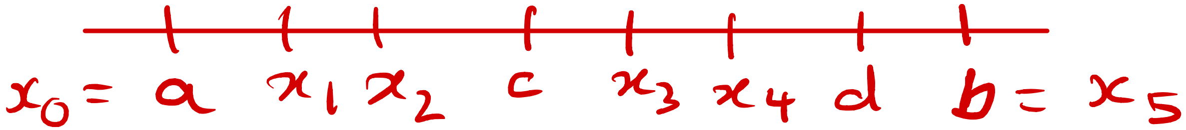
$$P = \{a, x_1, x_2, x_3, x_4, b\}$$



$$P' = \{a, x_1, x_2, c, x_3, x_4, d, b\}$$

$$= \{y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$$

P' refinement of P.

P'' $\{y_3, y_4, y_5, y_6\}$ = Partition of $[c,d]$

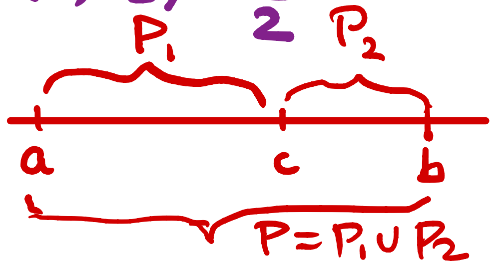
Thm 7.1.2 Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,c]$ and $[c,b]$, $c\in(a,b)$, then

$$\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^b f(x)\,dx \qquad \int_a^c f+\int_c^b f=\int_a^b f$$

Proof Let $\varepsilon>0$ & let P_1, P_2 be partitions of $[a,c]$ & $[c,b]$ respectively such that

$$U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2},\quad U(f,P_2)-L(f,P_2)<\frac{\varepsilon}{2}$$

Let $P=P_1\cup P_2$, then P is a partition of $[a,b]$.



$$U(f,P)-L(f,P)=U(f,P_1)+U(f,P_2)-L(f,P_1)-L(f,P_2)$$

$$=\big(U(f,P_1)-L(f,P_1)\big)+\big(U(f,P_2)-L(f,P_2)\big)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

$\therefore f$ is Riemann integrable on $[a,b]$.

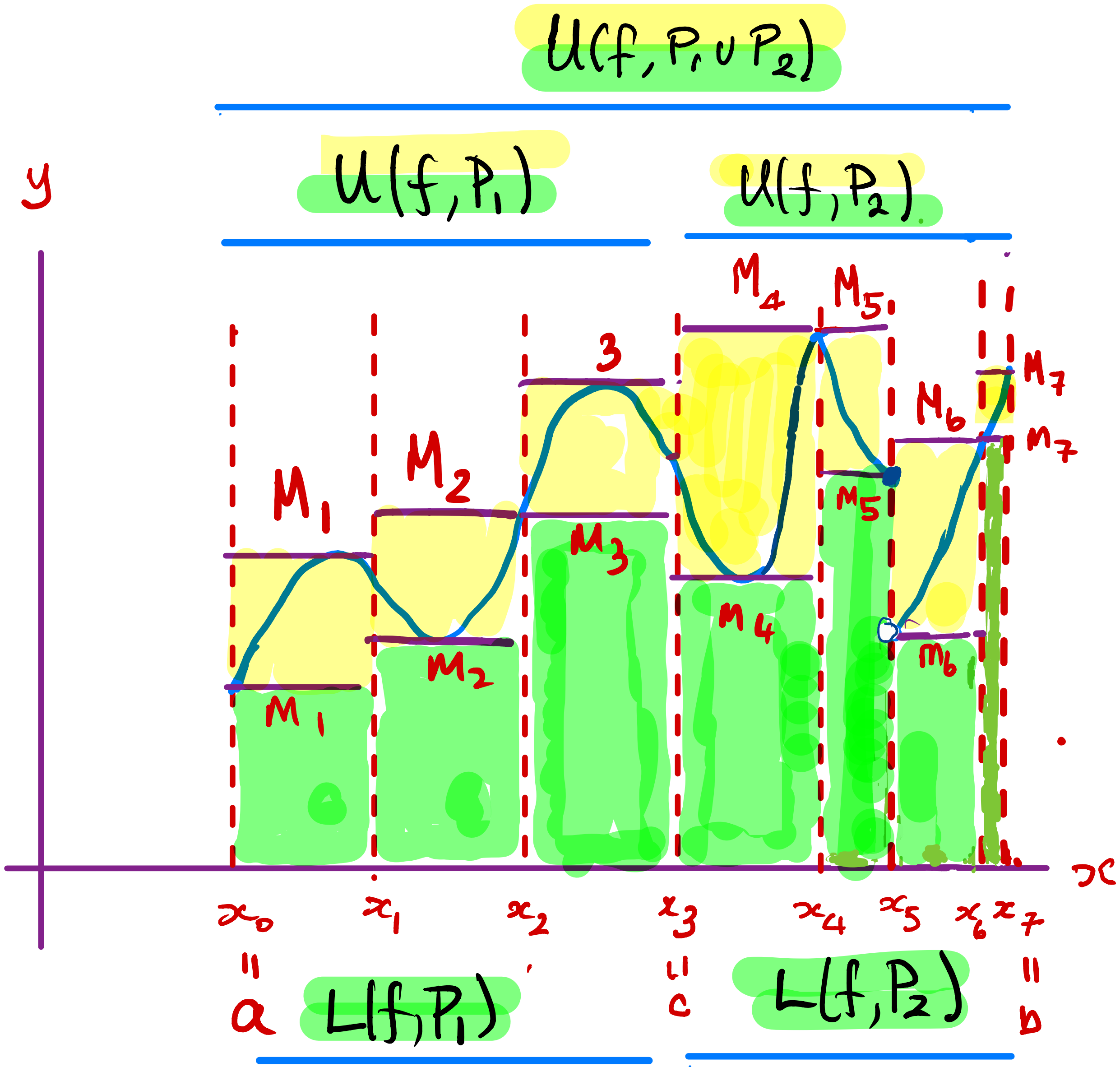


Illustration of LOWER/UPPER APP SUMS for $\int_a^c f + \int_c^b f$

$$P_1 = \{x_0, x_1, x_2\ x_3\} \quad P_2 = \{x_3, x_4, x_5, x_6\} \rightarrow P = P_1 \cup P_2$$

$$L(f,P_1) \le \int_a^c f \le U(f,P_1); \qquad L(f,P_2) \le \int_c^b f \le U(f,P_2)$$

$$\Rightarrow \quad L(f,P) = L(f,P_1) + L(f,P_2) \le \int_a^c f + \int_c^b f \le U(f,P_1) + U(f,P_2) = U(f,P)$$

So we have

$$L(f,P) \le \int_a^c f + \int_c^b f \le U(f,P). \qquad (1)$$

$$L(f,P) < \int_a^b f < U(f,P)$$

$$L(f,P) \le \int_a^b f \le U(f,P) \qquad (2)$$

$$\Downarrow$$

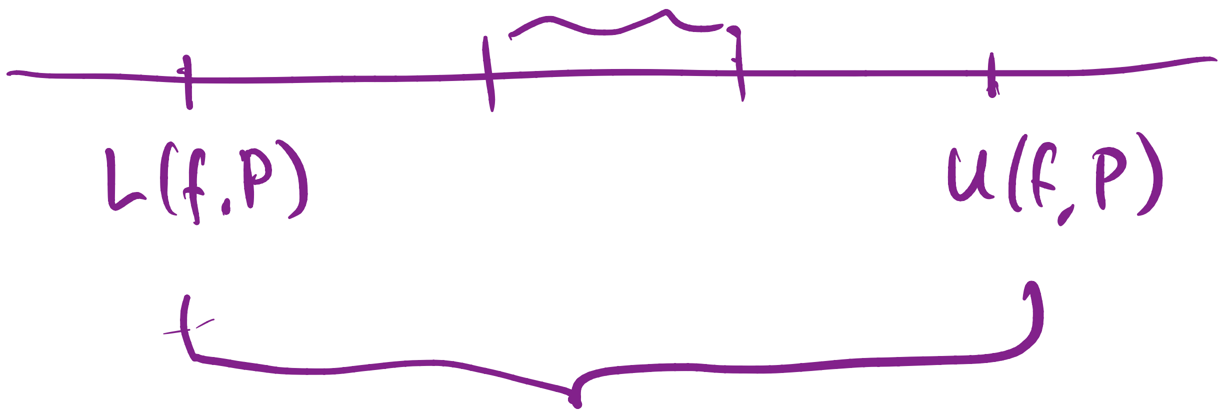
$$-U(f,P) \le -\int_a^b f \le -L(f,P) \qquad (2')$$

Adding (1) and (2') gives

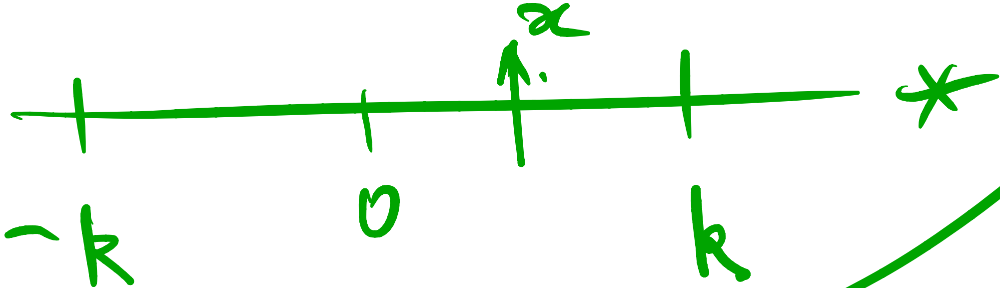
$$L(f,P) - U(f,P) \le \int_a^c f + \int_c^b f - \int_a^b f \le U(f,P) - L(f,P)$$

$$\overset{\shortparallel}{-(U(f,P) - L(f,P))} \le \quad " \quad \le U(f,P) - L(f,P)$$

$\int_a^c f + \int_c^b f \qquad \int_a^b f$



$< \varepsilon$



$|x| < k.$

$$\left|\int_a^c f + \int_c^b f - \int_a^b f\right| < U(f,P) - L(f,P) < \varepsilon$$

$$\Rightarrow \quad \int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Lemma 7.1.3 Let $f, g : [a,b] \to \mathbb{R}$ be bounded and let P be a partition of $[a,b]$,

i) $U(f+g, P) \leq U(f,P) + U(g,P)$

ii) $L(f+g, P) \geq L(f,P) + L(g,P)$

$(f+g)(x) = f(x)+g(x)$

Proof For a subinterval I of P we write for function h

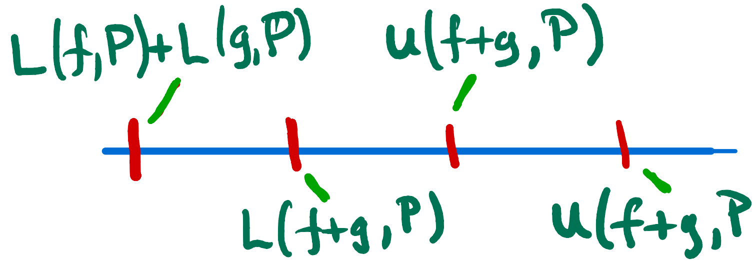
$$M(h) = \sup\{h(x),\ x \in I\}$$

$$m(h) = \inf\{h(x),\ x \in I\}$$

$$M(f+g) = \sup\{f(x)+g(x) \mid x \in I\}$$

$$\le \sup\{f(x) \mid x \in I\} + \sup\{g(x) \mid x \in I\}$$

$$\le M(f) + M(g)$$



L

$$U(f+g,P) = \sum_{i=1}^{n} \overset{m_i}{M_i}(f+g)(x_i - x_{i-1})$$

$$\underset{\ge}{\le} \sum_{i=1}^{n} \overset{m_i}{M_i}(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} \overset{m_i}{M_i}(g)(x_i - x_{i-1})$$

$$= U(f,P) + U(g,P)$$

<u>A similar argument holds for $L(f+g,P)$</u> (changes in red)

Theorem 7.1.4 Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable, then $f+g$, cf are Riemann integrable $\forall c \in \mathbb{R}$

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b (f(x)+g(x))\,dx \quad \&$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Proof Using Lemma 7.1.3

Let $\varepsilon > 0$ and $\exists$ partitions P_1 and P_2 such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad \& \quad U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$ then

$$U(f,P) - L(f,P) \leq U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$$

$$U(g,P) - L(g,P) \leq U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}$$

and

$$U(f+g,P) - L(f+g,P)$$

$$\leq U(f,P) + U(g,P) - L(f,P) - L(g,f) < \varepsilon$$

$\therefore$ $f+g$ is Riemann integrable on $[a,b]$.

To show $\int f + \int g = \int f+g$,

Consider

$$\overset{L}{\overset{\shortparallel}{L(f,P)+L(g,P)}} \le \int_a^b f + \int_a^b g \le \overset{U}{\overset{\shortparallel}{U(f,P)+U(g,P)}} \quad ①.$$

Also

$$\overset{L}{\overset{\shortparallel}{L(f,P)+L(g,P)}} \le L(f+g,P)$$

$$\le \int_a^b (f+g) \le U(f+g,P)$$

$$\le \overset{U}{\overset{\shortparallel}{U(f,P)+U(g,P)}} \quad ②$$

$$2 \Rightarrow 2' \quad -U \leq -\int_a^b (f+g) \leq -L \qquad (2 \times -1)$$

$$1 \quad L \leq \int_a^b f + \int_a^b g \leq U \qquad (\text{copy above})$$

ADD 1 and 2':

$$-U + L \leq \int_a^b f + \int_a^b g - \int_a^b f+g \leq U - L$$

$-(U-L)$

$$\left| \int_a^b f + \int_a^b g - \int_a^b (f+g) \right| < U - L < \varepsilon$$

$$\therefore \boxed{\int_a^b (f+g) = \int_a^b f + \int_a^b g}$$

Or in usual notation:

$$\boxed{\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$$

The proof for "cf" is left as an exercise!