Mathematical Tools of Asset Management
MTH6113

Dr Melania Nica

1. Consider an economy where there are just two companies, ABS and BNA whose shares are available for investment. The expected return on ABS shares is 8%, and the expected return on BNA shares is 12%. The rates of return of these two stocks have a correlation coefficient of 0.2. The standard deviation of the rates of return on Abacus's shares is 4% and the standard deviation of the return on Banana's shares is 8%. An investor prefers more to less and can short sell both assets (i.e. hold negative amounts of either asset).

    1. A portfolio P is formed using only ABS and BNA shares and which has the lowest global variance. Derive P and hence calculate the expected return and the standard deviation for this portfolio.

    For minimum variance portfolio, P, invest proportion w_A in ABS shares with

    $w_A = \frac{\sigma_B^2 - \rho\sigma_A\,\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\,\sigma_B} = \frac{0.0064 - 0.2\times 0.04\times 0.08}{0.0016 + 0.0064 - 2\times 0.2\times 0.04\times 0.08} = \frac{6}{7}$ and

    Proportion invested in BNA shares$= \frac{1}{7}$

    $E_P = \frac{6}{7}\times 0.08 + \frac{1}{7}\times 0.12 = 0.08571$ or 8.571%

    $V_P = \left(\frac{6}{7}\right)^2 \times 0.0016 + \left(\frac{1}{7}\right)^2 \times 0.0064 + 2\times\frac{6}{7}\times\frac{1}{7}\times 0.2\times 0.04\times 0.08 = 0.0014629$

    $\sigma_P = 0.03825 = 3.825\%$

2. Suppose the investor requires an expected return of 9% with the lowest possible variance from a portfolio Q formed using ABS and BNA shares. Derive Q and hence calculate the standard deviation for this portfolio.

If $E_Q = 9\%$ then Lagrangian is:

$W = V + \lambda\left(E_Q - E\right) + \mu(1 - \Sigma w_i)$

Set

$$\frac{\partial W}{\partial \lambda} = E_Q - \sum_i E_i w_i = 0.09 - 0.08 w_A - 0.12 w_B = 0$$
$$\frac{\partial W}{\partial \mu} = 1 - \sum_i w_i = 1 - w_A - w_B = 0$$

Solving gives $w_A = 0.75, w_B = 0.25$

$$\sigma_Q^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2\rho w_A w_B \sigma_A \sigma_B$$

$=0.75^2(0.04)^2+0.25^2(0.08)^2+2 \times 0.2 \times 0.75 \times 0.25 \times 0.04 \times 0.08 = 0.00154$

So $\sigma_Q = 0.0392 = 3.92\%$.

3. Derive the general equation for the efficient frontier.

$$E = w_A E_A + w_B E_B = 0.08 w_A + 0.12(1 - w_A) \Rightarrow w_A = \frac{0.12 - E}{0.04}, w_B = \frac{E - 0.08}{0.04}$$

$$\sigma^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2\rho w_A w_B \sigma_A \sigma_B$$

$$\sigma^2 = \left(\frac{0.12 - E}{0.04}\right)^2 0.0016 + \left(\frac{E - 0.08}{0.04}\right)^2 0.0064 + 2 \times 0.2 \left(\frac{0.12 - E}{0.04}\right)\left(\frac{E - 0.08}{0.04}\right) \times 0.04 \times 0.08$$

$$0 = 4.2E^2 - 0.72E - 0.0368 - \sigma^2$$

Thus the efficient frontier is:

$$E = \frac{1}{2 \times 4.2}(0.72 + \sqrt{(0.72)^2 + 4 \times 4.2 \times (0.0368 + \sigma^2})$$

Ignore inefficient part which is:

$$E = \frac{1}{2 \times 4.2}(0.72 - \sqrt{(0.72)^2 + 4 \times 4.2 \times (0.0368 + \sigma^2})$$

2. Assume that in an economy there are only two assets available for investment, A and B. The expected returns (as percentages), denoted by E, are $E_A = 9\%$ and $E_B = 6\%$ respectively. The risks, measured by standard deviations, denoted by σ, are $\sigma_A = 20\%$ and $\sigma_B = 10\%$ respectively. The returns on the assets are perfectly negatively correlated.

1.

(i) Derive the general expression for the mean and standard deviation of a portfolio formed from assets A and B.

(ii) Hence construct a risk-free portfolio F formed from assets A and B.

A portfolio F formed from assets A and B will have the following returns:

$R_F = w_A R_A + (1 - w_A) R_B$

with

$Var(R_F) = \sigma_F^2 = w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2\rho w_A w_A \sigma_A \sigma_B$ and with $\rho = -1$
$= w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 - 2 w_A w_A \sigma_A \sigma_B$
$= (w_A \sigma_A - (1 - w_A)\sigma_B)^2$

and $\sigma_F = w_A \sigma_A - (1 - w_A)\sigma_B$

But risk-free portfolio has zero variance and so $w_A \sigma_A = (1 - w_A)\sigma_B$

$$\Rightarrow w_A = \frac{\sigma_B}{\sigma_A+\sigma_B} = \frac{0.10}{0.30} = \frac{1}{3}$$

So risk-free portfolio, F, is formed of $\frac{1}{3}$A and $\frac{2}{3}$B.

2.

(i) Derive the equation of the efficient frontier for the economy. (Note: A will on efficient frontier as it is asset with highest expected return but B may not be on efficient frontier as it may lie below point of minimum variance).

$$E_F = w_A E_A + (1 - w_A)E_B = \frac{1}{3}0.09 + \frac{2}{3}0.06 = 0.07$$

A and B are perfectly negatively correlated and so efficient frontier is a straight line through A and F (note: A is on efficient frontier as it is asset with highest expected return but can't guarantee B is on efficient frontier).

Equation of efficient frontier is therefore $\frac{E-0.09}{\sigma-0.20} = \frac{0.07-0.09}{0-0.20}$ which simplifies to $E = 0.07 + \frac{1}{10}\sigma$

(ii) Hence construct a portfolio P formed from assets A and B which has the same risk as B but a higher expected return.

On the efficient frontier, $\sigma = 0.10 \Rightarrow E = 0.08$

This portfolio is halfway between A and F on the efficient frontier and so is made up of of $\frac{1}{2}$A and $\frac{1}{2}$F.

Given the construction of F, this equates to $\frac{2}{3}$A and $\frac{1}{3}$B.

c)

i) Draw and label the graph of the opportunity set on the $E-\sigma$ space, showing the positions of A, B, F, P and the efficient frontier.

ii) Discuss the relationship between an investor's indifference curves and risk/return preference.