It may occur that the samples will depend on a common parameter, but have different distributions.

Example

The number of claims X from a low risk policy is Poisson(μ), while the number of claims from a high risk policy is Poisson(2μ).

A sample of 15 low risk policies had a total of 48 claims.
A sample of 10 high risk policies had a total of 59 claims.

$$L(\mu; \underline{x}, \underline{y}) = \prod_{i=1}^{15} P(X = x_i) \prod_{i=1}^{10} P(Y_i = y_i)$$

$$\sum_{i=1}^{15} x_i = 48$$

$$\sum_{i=1}^{10} y_i = 59$$

$$L(\mu; \underline{x}, \underline{y}) = \prod_{i=1}^{15} \frac{e^{-\mu}\mu^{x_i}}{x_i!} \prod_{i=1}^{10} \frac{e^{-2\mu}(2\mu)^{y_i}}{(y_i)!}$$

$$= \frac{e^{-15\mu} \mu^{\sum_{i=1}^{15} x_i}}{\prod_{i=1}^{15} x_i!} \; \frac{e^{-20\mu} \, 2^{\sum_{i=1}^{10} y_i} \mu^{\sum_{i=1}^{10} y_i}}{\prod_{i=1}^{10} y_i!}$$

$$= e^{-35\mu} \mu^{48+59} \times f(\underline{x}, \underline{y})$$

$$= e^{-35\mu} \mu^{107} f(\underline{x}, \underline{y})$$

$$\ell(\mu; \underline{x}, \underline{y}) = -35\mu + 107 \ln\mu + \ln f(\underline{x}, \underline{y})$$

$$\frac{\partial \ell}{\partial \mu} = -35 + \frac{107}{\mu} = 0$$

$$\hat{\mu} = \frac{107}{35} = 3.057$$

$$\frac{\partial^2 \ell}{\partial \mu^2} = -\frac{107}{\mu^2} < 0$$ so we found MLE.

Topics for simulating random variables

The Gamma Distributions

The Gamma family of distributions has two parameters, either

α shape parameter
λ rate parameter

or α shape parameter
$s = \frac{1}{\lambda}$ scale parameter

all parameters are positive

We write Gamma(α, λ) or $\Gamma(\alpha, \lambda)$

Gamma distributions are continuous and positive.

The Gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt$$

We have $\Gamma(1) = 1$

Integration by parts gives

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

so $\Gamma(2) = 1 \cdot \Gamma(1) = 1$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2$$

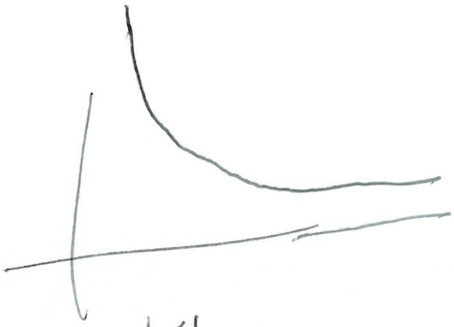
$$\vdots$$

$$\Gamma(n) = (n-1)!$$

We also have $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$.

The p.d.f. of a $\Gamma(\alpha, \lambda)$ random variable

is $f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\lambda)} y^{\alpha-1} e^{-\lambda y}$ for $y > 0$

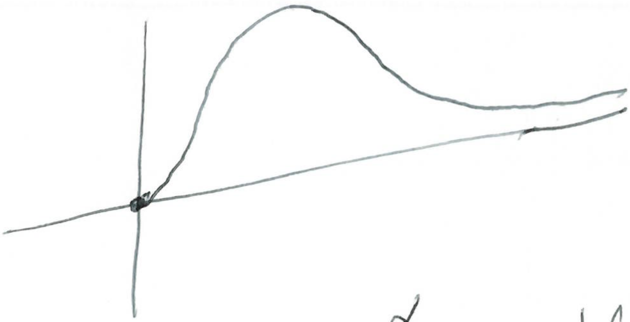


$\alpha < 1$

$\alpha = 1$

Exponential

$\alpha > 1$



$$E(Y) = \frac{\alpha}{\lambda} \qquad Var(Y) = \frac{\alpha}{\lambda^2}$$

The Beta Distribution

The Beta Family of distribution has two parameters $\alpha > 0$, $\beta > 0$

Beta distributed random variables are continuous and take values in the interval $[0,1] = \{x \in \mathbb{R} : 0 \le x \le 1\}$.

The Beta function is defined by

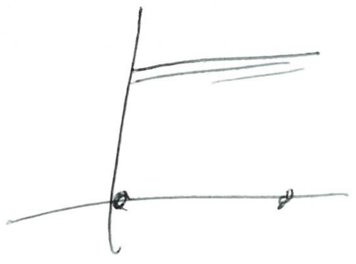
$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

The relationship between beta and gamma functions is

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

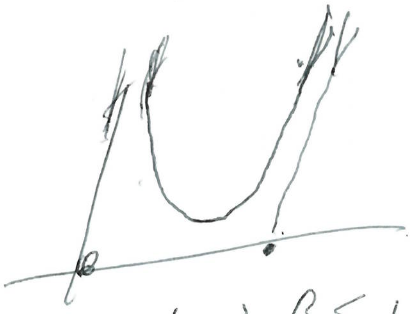
The p.d.f. of the $B(\alpha,\beta)$ distribution is

$$f_Y(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \quad \text{for } 0 \le y \le 1$$

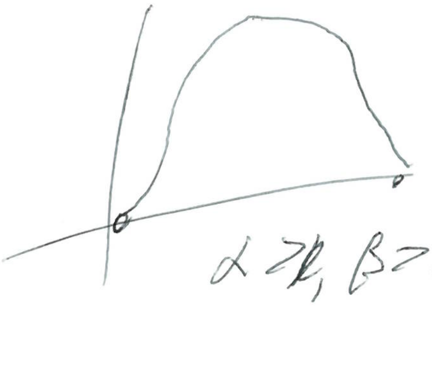


$\alpha=\beta=1$

Uniform $[0,1]$



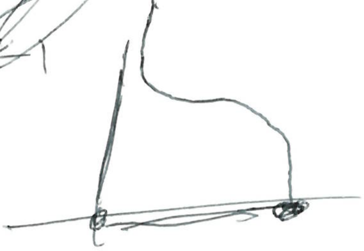
$\alpha<1, \beta<1$



$\alpha>1, \beta>1$

not symmetric if $\alpha \neq \beta$

~~$\alpha>1, \beta<1$~~



$\alpha<1, \beta>1$

$$E(Y) = \frac{\alpha}{\alpha+\beta} \qquad Var(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$E(Y) = \int_0^1 y \, \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} \, dy$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 y^{\alpha+1-1}(1-y)^{\beta-1} \, dy$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

The Log-Normal Distribution

The LogNormal family of distributions have two parameters μ and $\sigma^2 > 0$.

We write LogNormal(μ, σ^2),

A r.v. $Y \sim$ LogNormal(μ, σ^2) has the distribution $Y = e^Z$ where $Z \sim N(\mu, \sigma^2)$

It follows that Y is continuous and positive.

We have

$$P(Y \le y) = P(e^Z \le y) = P(Z \le \ln y)$$

$$= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$\Rightarrow f_Y(y) = \frac{d}{dy} P(Y \le y)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln y-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{y}$$

As $y \to \infty$ f_Y decreas like

$$e^{-c \ln y^2} \cdot \frac{1}{y}$$

much slower that $\underbrace{e^{-\lambda y} y^{\alpha-1}}_{\text{Gamma}}$

The lognormal distributions have very long tails.

$$E(Y) = e^{\mu + \frac{\sigma^2}{2}}$$

$$E(Y^2) = e^{2\mu + 2\sigma^2}$$

$$Var(Y) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$= e^{2\mu}\left(e^{2\sigma^2} - e^{\sigma^2}\right)$$

Simulating random variable using the uniform distribution

Suppose that you want to simulate a continuous random variable with c.d.f. F. We will show how to generate a r.v. with c.d.f. F using the Uniform[0, 1] distribution.

Note: F^{-1} denotes inverse function not $1/F$

Lemma

If $U \sim \text{Uniform}[0,1]$

then $F^{-1}(U)$ is a random variable with c.d.f. F