(4) a) We use a direct computation and the chain rule:

$$\frac{\partial^2 \phi}{\partial x'^a \partial x'^b} = \frac{\partial}{\partial x'^a}\left(\frac{\partial x^k}{\partial x'^b}\frac{\partial \phi}{\partial x^k}\right) = \frac{\partial x^l}{\partial x'^a}\frac{\partial}{\partial x^l}\left(\frac{\partial x^k}{\partial x'^b}\frac{\partial \phi}{\partial x^k}\right)$$

$$= \frac{\partial x^l}{\partial x'^a}\left(\frac{\partial^2 x^k}{\partial x^l \partial x'^b}\right)\frac{\partial \phi}{\partial x^k} + \frac{\partial x^l}{\partial x'^a}\frac{\partial x^k}{\partial x'^b}\frac{\partial^2 \phi}{\partial x^l \partial x^k}$$

The second term on the r.h.s. is the expression that we would expect for the transformation of a $(0,2)$ tensor. However, the first term is not and hence we conclude that $\frac{\partial^2 \phi}{\partial x^a \partial x^b}$ does NOT transform as a $(0,2)$ tensor.

b) Notice that if $V_{ab} = V_{ba}$, then

$$V_{ab} - V_{ba} = 0$$

Under coordinate transformations,

$$V'_{ab} - V'_{ba} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} V_{cd} - \frac{\partial x^e}{\partial x'^b} \frac{\partial x^f}{\partial x'^a} V_{ef}$$

$$= \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} V_{cd} - \frac{\partial x^f}{\partial x'^a} \frac{\partial x^e}{\partial x'^b} V_{fe}$$

$$= 0$$

where in the second line we have used that $V_{ef} = V_{fe}$ and in the third line we used that the indices (c,d) and (e,f) are dummy indices.

⑤ a) We are given

$$x^1 = x = e^\rho \cos\theta$$

$$x^2 = y = e^\rho \sin\theta$$

The inverse relations are:

$$x'^1 = \rho = \frac{1}{2}\ln(x^2+y^2)$$

$$x'^2 = \theta = \arctan\left(\frac{y}{x}\right)$$

Now we can calculate $\frac{\partial x'^a}{\partial x^b}$ and $\frac{\partial x^a}{\partial x'^b}$. We find:

$$\frac{\partial x'^1}{\partial x^1} = \frac{\partial \rho}{\partial x} = x e^{-2\rho} = e^{-\rho}\cos\theta, \quad \frac{\partial x'^1}{\partial x^2} = \frac{\partial \rho}{\partial y} = y e^{-2\rho} = e^{-\rho}\sin\theta$$

$$\frac{\partial x'^2}{\partial x^1} = \frac{\partial \theta}{\partial x} = -y e^{-2\rho} = -e^{-\rho}\sin\theta, \quad \frac{\partial x'^2}{\partial x^1} = \frac{\partial \theta}{\partial y} = x e^{-2\rho} = e^{-\rho}\cos\theta$$

Similarly,

$$\frac{\partial x^1}{\partial x'^1} = \frac{\partial x}{\partial \rho} = e^\rho\cos\theta = x, \quad \frac{\partial x^1}{\partial x'^2} = \frac{\partial x}{\partial \theta} = -e^\rho\sin\theta = -y$$

$$\frac{\partial x^2}{\partial x'^1} = \frac{\partial y}{\partial \rho} = e^\rho\sin\theta = y, \quad \frac{\partial x^2}{\partial x'^2} = \frac{\partial y}{\partial \theta} = e^\rho\cos\theta = x$$

(7) $A^{ab} = A^{ba}$, $B_{ab} = -B_{ba}$

$$\Rightarrow A^{ab} B_{ab} = - A^{ab} B_{ba} = - A^{ba} B_{ba} = - A^{ab} B_{ab}$$

The first equality follows from the antisymmetry of B_{ab}, the second equality follows from the symmetry of A^{ab}, and the third equality follows from the fact that the indices (a,b) are dummy indices and we can re-label them at our convenience.

Since $A^{ab} B_{ab} = - A^{ab} B_{ab}$ it follows that $A^{ab} B_{ab} = 0$

① Question: Geometry

- Consider the object $F_{ij} = \partial_i A_j - \partial_j A_i$, where A_i is a $(0,1)$ tensor. Compute the transformation of F_{ij} under a change of coordinates. $x^a = x^a(x'^b)$. Is F_{ij} a tensor?

- Solution:

Since A_i is a tensor, it transforms as

$$A_i = \frac{\partial x'^j}{\partial x^i} A'_j$$

As for the partial derivatives one has,

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j}$$

Putting these two results together, we get

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i \\ &= \frac{\partial x'^k}{\partial x^i} \frac{\partial}{\partial x'^k} \left(\frac{\partial x'^l}{\partial x^j} A'_l \right) - \frac{\partial x'^p}{\partial x^j} \frac{\partial}{\partial x'^p} \left(\frac{\partial x'^q}{\partial x^i} A'_q \right) \\ &= \frac{\partial x'^k}{\partial x^i} \left[\frac{\partial^2 x'^l}{\partial x'^k \partial x^j} A'_l + \frac{\partial x'^l}{\partial x^j} \frac{\partial}{\partial x'^k} A'_l \right] \end{aligned}$$

$$- \frac{\partial x'^p}{\partial x^j}\left[\frac{\partial^2 x'^q}{\partial x'^p \partial x^i} A'_q + \frac{\partial x'^q}{\partial x^i}\frac{\partial}{\partial x'^p} A'_q \right]$$

The first terms on each line cancel. To see this, one realises that they are equal to

$$\frac{\partial x'^k}{\partial x^i}\frac{\partial}{\partial x'^k}\left(\frac{\partial x'^l}{\partial x^j}\right) A'_l = \frac{\partial^2 x'^l}{\partial x^i \partial x^j} A'_l$$

$$\frac{\partial x'^p}{\partial x^j}\frac{\partial}{\partial x'^p}\left(\frac{\partial x'^q}{\partial x^i}\right) A'_q = \frac{\partial^2 x'^q}{\partial x^j \partial x^i} A'_q$$

Therefore, one is left with,

$$F_{ij} = \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j}\frac{\partial A'_l}{\partial x'^k} - \frac{\partial x'^p}{\partial x^j}\frac{\partial x'^q}{\partial x^i}\frac{\partial}{\partial x'^p} A'_q$$

$$= \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j}\left(\frac{\partial A'_l}{\partial x'^k} - \frac{\partial A'_k}{\partial x'^l}\right)$$

$$= \frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j} F'_{lk}$$

Therefore, F_{ij} transforms as a $(0,2)$ tensor.

(11) b) By the definition of covariant derivative,

$$\nabla_i B_{jk} = \partial_i B_{jk} - \Gamma^l_{ij} B_{lk} - \Gamma^l_{ik} B_{jl}$$

c) $\nabla_{[i} B_{jk]} = \partial_{[i} B_{jk]}$

since all the indices are anti-symmetric while $\Gamma^l_{ij} B_{lk}$ and $\Gamma^l_{ik} B_{jl}$ are symmetric in the pairs of indices (ij) and (ik) respectively.

d) Since $\partial_{[i} B_{jk]} = \nabla_{[i} B_{jk]}$ and the RHS is clearly a tensor, so must be the LHS.