Recap quiz (paraphrased defns/theorems)

Consider an LP in standard equation form

$$\begin{aligned} &\text{maximize} \quad \underline{c}^T \underline{x} \\ &\text{subject to} \quad A\underline{x} = \underline{b}, \quad \underline{x} \geq 0. \end{aligned}$$

A basic feasible solution is a ________ solution $\underline{x}$ in which the ________ entries of $\underline{x}$ correspond to ________ columns of A.

Last time we proved two results

1. Every LP (in standard equation form) has an ________ solution that is an ________ solution (provided it has at least one ________ solution).

2. Given an LP in standard equation form every ________ solution is an ________ solution and vice versa.

(proof not completed)

1 + 2 imply

__Corollary__ If an LP has an optimal solution, then it also has an optimal solution that is also a basic feasible solution.

Paste

A basic feasible solution is a ________ solution $\underline{x}$ in which the ________ entries of $\underline{x}$ correspond to __________ columns of A.

<u>Defn</u> For a basic feasible solution $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ of an LP in standard equation form

the <u>basic variables</u> are those x_i that are non-zero

the <u>non-basic variables</u> are those x_i that are zero.

Summary

- Start with a BFS
- At each step find a BFS with larger objective value by increasing one variable from 0 and decreasing one variable to 0.
- Rewrite LP so it becomes obvious which variable to increase in the next step
- Stop when we see that we cannot increase the objective function my move.

Systematic description of simplex algoritum

Simpler case when $\underline{b} \geq \underline{0}$.

Assume you are given an LP in standard inequality form

$$\begin{aligned} &\text{maximise} \quad \underline{c}^T \underline{x} \\ &\text{subject to} \quad A\underline{x} \leq \underline{b} \quad \underline{x} \geq \underline{0}. \end{aligned}$$

Here A is $m \times n$ matrix, $\underline{c} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$, $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$
(so m constraints, n variables)

① Initialisation:

Put in standard equation form by introducing m slack variables $s_1, \ldots, s_m$.

$$\begin{aligned} &\text{maximise} \quad \underline{c}'^T \underline{x}' \\ &\text{subjet to} \quad A'\underline{x}' \leq \underline{b}, \quad \underline{x}' \geq \underline{0} \end{aligned}$$

$A' = (A \mid I)$ $m \times (n+m)$ matrix

$\underline{c}'^T = (\underline{c} \mid \underline{0}) \in \mathbb{R}^{n+m}$ (add m zeros to $\underline{c}^T$)

$\underline{x}' = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_m \end{pmatrix} \in \mathbb{R}^{n+m}$

Construct initial tableau (see example)

Basic variables

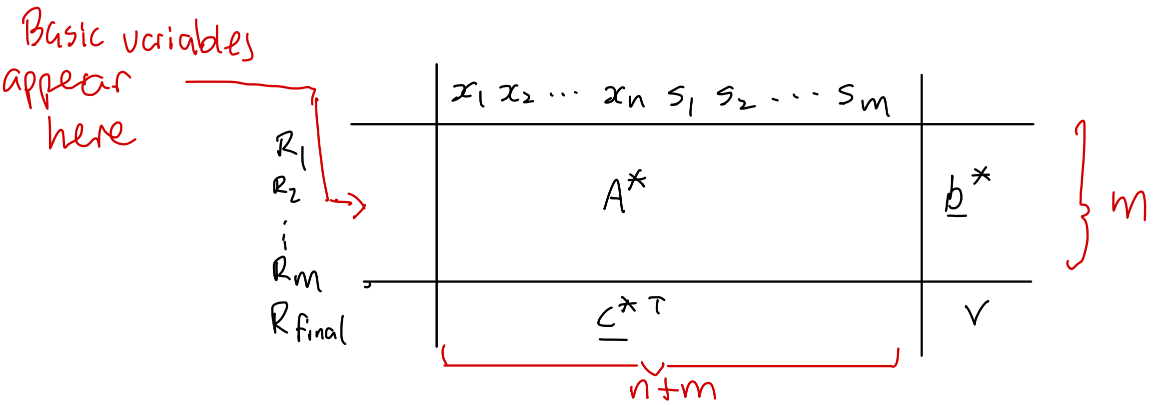
(to begin the basic variables are the slack variables)

	$x_1 \; x_2 \cdots x_n \; s_1 \; s_2 \cdots s_m$	
s_1, s_2, $\vdots$, s_m	$A' = (A \mid I)$	$\underline{b}$
	$c'^T = (\underline{c}^T \mid \underline{0})$	0

$\}\, m$

$\underbrace{\qquad\qquad}_{n+m}$

② Repeatedly apply pivot steps as follows.
Consider current tableux



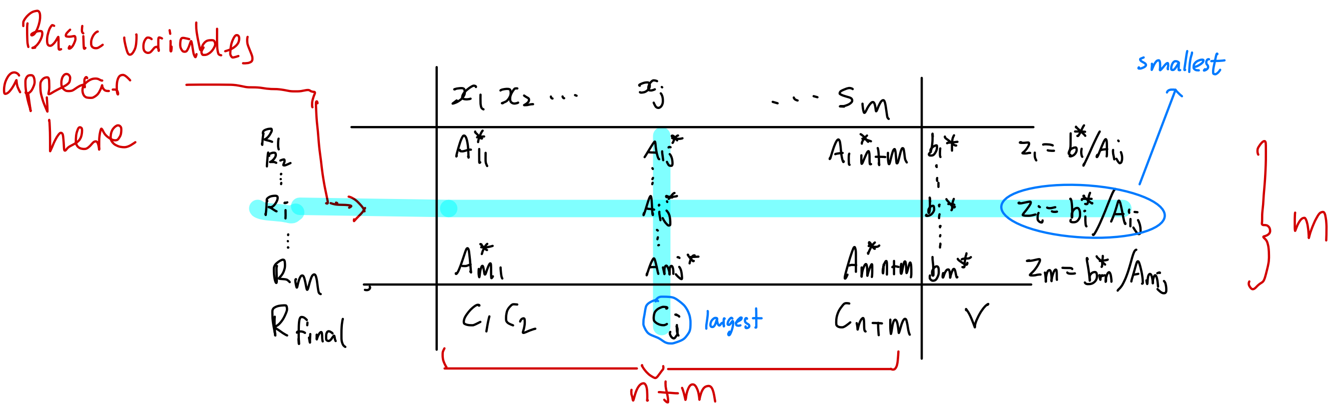
Label rows $R_1, R_2, \dots, R_m, R$ (just so we can refer to them)

(a) Find largest positive entry in $\underline{c}^{*T}$, say c_j^*, and highlight j^{th} column

(b) Look at each entry in highlighted column i.e. the entries A_{rj}^* $r=1,\dots,m$

For each $r=1,\dots,m$ if $A_{rj}^* \geq 0$ let $z_r = b_r^*/A_{rj}^*$ and record this number z_r next to b_r^*

Of all z_r, pick smallest, say z_i, and highlight its row, i.e. R_i



(c) We "clear" j^{th} column (i.e. highlighted column) using row operations

- Replace i^{th} row R_i (i.e. highlighted row) by $R_i' = R_i/A_{ij}^*$ (so ij^{th} entry is now 1)
- Replace every other row R_r with $R_r' = R_r - A_{rj}^* R_i'$ (including R_{final})
  So all entries in highlighted column become zero except ij^{th}.
- Replace highlighted row variable with highlighted column variable (keeping all other variables unchanged).
- Now have our new tableau

(3) Repeat (2) until either

(a) c^{*T} has no positive entries in step 2(a).
In this case the optimal solution is obtained by setting each variable on the far left to the value on the far right and all other variables to zero
The maximum objective value is the negative of the bottom right entry.

(b) There are no positive entries in the highlighted column in step 2(b).
If this happens the LP is unbounded

Important notes

- In each pivot,
  the variable at the top of highlighted column is called the entering variable
  the variable to left of highlighted row is called the leaving variable.

- Tie breaking rules.

When picking largest value in a row / smallest value in a column
If there is a tie-break, pick the one fartherest left / closest to the top.

- How would you summarise this for exam?!
- Roughly what is the reason for steps 2(a) 2(b) 2(c).

Apply simplex to following example

$$\text{maximise} \quad 4x_1 + \tfrac{1}{2}x_2$$

$$\text{sub to} \quad \begin{aligned} x_1 + x_2 &\leq 3 \\ \tfrac{1}{2}x_1 + x_2 &\leq 2 \\ \tfrac{1}{2}x_1 - x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

① Put in standard eqn form

$$\max \quad 4x_1 + \tfrac{1}{2}x_2$$

$$\text{sub to} \quad \begin{aligned} x_1 + x_2 + s_1 \quad\quad &= 3 \\ \tfrac{1}{2}x_1 + x_2 \quad + s_2 \quad &= 2 \\ \tfrac{1}{2}x_1 - x_2 \quad\quad + s_3 &= 1 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

$A' =$

Initial tableux

	x_1	x_2	s_1	s_2	s_3	
s_1	1	1	1	0	0	3
s_2	1/2	1	0	1	0	2
s_3	1/2	-1	0	0	1	1
	4	1/2	0	0	0	0

Pivot step 2

2a, 2b

		x_1	x_2	s_1	s_2	s_3		
R_1	s_1	1	1	1	0	0	3	$3/1 = 3$
R_2	s_2	$1/2$	1	0	1	0	2	$2/\tfrac{1}{2} = 4$
R_3	s_3	$1/2$	-1	0	0	1	1	$\tfrac{1}{2}/1 = \tfrac{1}{2}$
R_{final}		4	$1/2$	0	0	0	0	

2c

		x_1	x_2	s_1	s_2	s_3	
$R_1' = R_1 - R_3'$	s_1	0	3	1	0	-2	1
$R_2' = R_2 - \frac{1}{2}R_3'$	s_2	0	2	0	1	-1	1
$R_3' = R_3/\tfrac{1}{2}$	x_1 ~~s_3~~	1	-2	0	0	2	2
$R_{final}' = R_{final} - 4R_3'$		0	$17/2$	0	0	-8	-8

Pivot again with updated table

Apply pivot to updated tableax.

		x_1	x_2	s_1	s_2	s_3		
R_1	s_1	0	3	1	0	-2	1	$1/3$
R_2	s_2	0	2	0	1	-1	1	$1/2$
R_3	x_1	1	-2	0	0	2	2	—
R_{final}		0	$17/2$	0	0	-8	-8	

		x_1	x_2	s_1	s_2	s_3	
$R_1' = \frac{1}{3}R_1$	x_2 ~~s_1~~	0	1	$1/3$	0	$-2/3$	$1/3$
$R_2' = R_2 - 2R_1'$	s_2	0	0	$-2/3$	1	$1/3$	$1/3$
$R_3' = R_3 + 2R_1'$	x_1	1	0	$2/3$	0	$2/3$	$8/3$
$R_{final}' = R_{final} - \frac{17}{2}R_1'$		0	0	$-17/6$	0	$-\frac{7}{3}$	$-\frac{65}{6}$

We do not apply another pivot because 3(a) tells us we have found an optimal solution.

Optimal solution

$$\begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 1/3 \\ 0 \\ 2/3 \\ 0 \end{pmatrix}$$

obj value for this optimal solution is $-\frac{65}{6}$

Example

maximise $2x_1 - x_2 + 8x_3$

subject to

$$2x_3 \leq 1$$
$$2x_1 - 4x_2 + 6x_3 \leq 3$$
$$-x_1 + 3x_2 + 4x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0$$

standard equation form

maximise $2x_1 - x_2 + 8x_3$

subject to

$$2x_3 + s_1 = 1$$
$$2x_1 - 4x_2 + 6x_3 + s_2 = 3$$
$$-x_1 + 3x_2 + 4x_3 + s_3 = 2$$
$$x_1, x_2, x_3 \geq 0.$$

Initial tableau

	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	0	0	2	1	0	0	1
s_2	2	-4	6	0	1	0	3
s_3	-1	3	4	0	0	1	2
	2	-1	8	0	0	0	0

		x_1	x_2	x_3	s_1	s_2	s_3		
R_1	s_1	0	0	2	1	0	0	1	$1/2$
R_2	s_2	2	-4	6	0	1	0	3	$3/6 = 1/2$
R_3	s_3	-1	3	4	0	0	1	2	$2/4 = \frac{1}{2}$
R_f		2	-1	8	0	0	0	0	

First pivot

		x_1	x_2	x_3	s_1	s_2	s_3	
$R_1' = \frac{1}{2}R_1$	x_3 $\cancel{s_1}$	0	0	1	$1/2$	0	0	$1/2$
$R_2' = R_2 - 3R_1$	s_2	2	-4	0	-3	1	0	0
$R_3' = R_3 - 2R_1$	s_3	-1	3	0	-2	0	1	0
$R_f' = R_f - 4R_1$		2	-1	0	-4	0	0	-4

		x_1	x_2	x_3	s_1	s_2	s_3		
R_1	x_3	0	0	1	1/2	0	0	1/2	—
R_2	x_1 ~~s_2~~	2	-4	0	-3	1	0	0	0
R_3	s_3	-1	3	0	-2	0	1	0	—
R_f		2	-1	0	-4	0	0	-4	

2^{nd} pivot

		x_1	x_2	x_3	s_1	s_2	s_3	
$R_1' = R_1$	x_3	0	0	1	1/2	0	0	1/2
$R_2' = \frac{1}{2}R_2$	x_1	1	-2	0	-3/2	1/2	0	0
$R_3' = R_3 + \frac{1}{2}R_2$	s_3	0	1	0	-7/2	1/2	1	0
$R_f' = R_f - R_2$		0	3	0	-1	-1	0	-4

		x_1	x_2	x_3	s_1	s_2	s_3		
R_1	x_3	0	0	1	$1/2$	0	0	$1/2$	—
R_2	x_1	1	-2	0	$-3/2$	$1/2$	0	0	—
R_3	x_2 $\cancel{s_3}$	0	1	0	$-7/2$	$1/2$	1	0	0
		0	3	0	-1	-1	0	-4	

3^{rd} pivot

		x_1	x_2	x_3	s_1	s_2	s_3	
$R_1' = R_1$	x_3	0	0	1	$1/2$	0	0	$1/2$
$R_2' = R_2 + 2R_3$	x_1	1	0	0	$-17/2$	$3/2$	0	0
$R_3' = R_3$	x_2	0	1	0	$-7/2$	$1/2$	1	0
$R_f' = R_f - 3R_3$		0	0	0	$19/2$	$-5/2$	-3	-4

			x_1	x_2	x_3	s_1	s_2	s_3		
R_1	s_1	~~x_3~~	0	0	1	$1/2$	0	0	$1/2$	$\frac{1/2}{1/2} = 1$
R_2		x_1	1	0	0	$-17/2$	$3/2$	0	0	—
R_3		x_2	0	1	0	$-7/2$	$1/2$	1	0	—
			0	0	0	$19/2$	$-5/2$	-3	-4	

		x_1	x_2	x_3	s_1	s_2	s_3	
$R_1' = 2R_1$	s_1	0	0	2	1	0	0	1
$R_2' = R_2 + 17R_1$	x_1	1	0	17	0	$3/2$	0	$17/2$
$R_3' = R_3 + 7R_1$	x_2	0	1	7	0	$1/2$	1	$7/2$
$R_f' = R_f - 19R_1$		0	0	-19	0	$-5/2$	-3	$-27/2$

all values ≤ 0
so algorithm stops

Optimal soln is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 17/2 \\ 7/2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ with obj value $-27/2$

Optimal soln to original LP is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 17/2 \\ 7/2 \\ 0 \end{pmatrix}$.