MTH6127 **Metric Spaces and Topology**

Course work 5 **23 February 2024**

1. Consider the Hamming metric on Σ^n in the alphabet $\Sigma = \{0, 1, 2\}$. What is the cardinality of the closed ball $B[w; 1]$?

   The ball $B[w; 1]$ contains the word w and all variations of w in a single position. Since there are n positions and each position can be varied by selection a symbol of Σ distinct from the current occupier of the position, we see that the cardinality of the ball $B[w; 1]$ is $2n + 1$.

2. Show that two open intervals $(a, b) \subset \mathbb{R}$ and $(a', b') \subset \mathbb{R}$ are isometric if and only if they have the same length, i.e. $b - a = b' - a'$.

   If $b - a = b' - a'$, the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + (a' - a)$ is an isometry mapping (a, b) onto (a', b'). Conversely, if the length of the interval (a, b) equals its diameter

   $$b - a = \sup\{d(x, y); x, y \in (a, b)\}.$$

   This quantity is clearly invariant under isometries, and thus two isometric intervals must have equal length.

3. Let $X = \mathbb{R}$ with the standard metric. Which of the following sets are dense in X?

   (a) The set A of rational numbers shifted by π, i.e. the set of numbers of the form $x = r + \pi$, where $r \in \mathbb{Q}$.

   This set A is dense: every open interval (a, b) contains a point of the form $r + \pi$ where $r \in \mathbb{Q}$. This is equivalent to the statement that any open interval $(a - \pi, b - \pi)$ contains a point $r \in \mathbb{Q}$.

   (b) The set B of rational multiples of $\sqrt{2}$, i.e. the set of numbers of the form $x = r \cdot \sqrt{2}$ where $r \in \mathbb{Q}$.

   This set B is dense. Indeed, $r\sqrt{2} \in (a, b)$ iff $r \in (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$. Since $\mathbb{Q}$ is dense, for any $a < b$ we can find $r \in \mathbb{Q}$ with $r \in (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$.

   (c) The set C of rational numbers whose decimal representation does not contain the digit "7".

   C is not dense as the open interval $(0.7, 0.71)$ contains no points of C.

4. Let (X, d) be a metric space. Let $Y \subset X$ be a finite subset. Prove that Y is closed.

   It is enough to show that a single point set $\{a\}$ is closed or that the set $\{a\}^c = \{x \in X; x \neq a\}$ is open. If $b \in \{a\}^c$ then $B(b; d(a, b)) \subset \{a\}^c$, i.e. $\{a\}^c$ is open.

5. Let $(V, ||\cdot||)$ be a normed space. Prove that the set $F = \{x \in V; ||x|| = 1\}$ is closed but not open.

   We show that the complement F^c is open. If $a \in F^c$, i.e. $||a|| \neq 1$ then $B(a;r) \subset F^c$ where $r = |(1 - ||a||)|$. This shows that F is closed.

   For $b \in F$, no open ball $B(b;r)$ is contained in F. Indeed, the ball $B(b;r)$ contains the point $(1+\epsilon)b$ for small $\epsilon > 0$ and $||(1+\epsilon)b|| = (1+\epsilon) \neq 1$.

6. Which of the following sets viewed with the metric induced from $\mathbb{R}$ are complete:

   (a) $(0, 1)$,

   Not complete as it is not closed.

   (b) $(0, \infty)$,

   Not complete as it is not closed.

   (c) $[0, \infty)$,

   Complete as it is closed.

   (d) $\mathbb{R} - \mathbb{Z}$,

   Not complete as it is not closed.

   (e) $\mathbb{Z}$,

   Complete as it is closed.

   (f) The Cantor set C,

   Complete as it is closed.

   (g) $\mathbb{Q}$.

   Not complete as it is not closed.