MTH6127 **Metric Spaces and Topology**

Course work 4 **15 February 2024**

1. In this question the set of real numbers $\mathbb{R}$ is equipped with the Euclidean metric. Which of the following sets are open or closed? Explain your answer.

   (a) $(0, 1)$,
   it is an open interval hence it is open.

   (b) $(0, \infty)$,
   it a union of open intervals $\cup_{n \geq 1}(0, n)$ hence it is open.

   (c) $[0, \infty)$,
   this set is not open as no open neighbourhood of 0 is contained in the set $[0, \infty)$. This set is closed as its complement $(-\infty, 0)$ is open as a union of open intervals.

   (d) $\mathbb{R} - \mathbb{Z}$,
   This set is open as it is the union of open intervals $\cup_{n \in \mathbb{Z}}(n, n+1)$. This set is not closed as its complement $\mathbb{Z}$ is not open.

   (e) $\mathbb{Q}$.
   This set is not open and not closed.

2. Let $f_0 : [0, 1] \to \mathbb{R}$ be any continuous function on $[0, 1]$. Denote the identity function by I, so that $I(x) = x$ for all x. Define the sequence (f_n) of functions in $C[0, 1]$ by

$$f_n(x) = \frac{1}{2}(f_{n-1}(x) + I(x)),$$

for all $n \geq 1$. Prove that

$$d(f_n, I) = \frac{1}{2} d(f_{n-1}, I)$$

where d is the sup-metric on $C[0, 1]$. Deduce that the sequence (f_n) converges in $(C[0, 1], d)$ to I.

If $f_n = \frac{1}{2}(f_{n-1} + I)$ then $f_n - I = \frac{1}{2}(f_{n-1} - I)$ and

$$d(f_n, I) = \sup_x (f_n(x) - I(x)) = \frac{1}{2} \sup_x (f_{n-1}(x) - I(x)) = \frac{1}{2} d(f_{n-1}, I).$$

We obtain by induction

$$d(f_n, I) = 2^{-n} d(f_0, I)$$

which shows that f_n converges to I as $n \to \infty$.

3. Consider the following metric d^* on $\mathbb{R}^2$:

$$d^*(p,q) = d^*\big((p_1,p_2),(q_1,q_2)\big) = \begin{cases} |p_1 - q_1|, & \text{if } p_2 = q_2; \\ |p_1 - q_1| + 1, & \text{otherwise.} \end{cases}$$

Which of the sets below are open in the metric space $(\mathbb{R}^2, d^*)$? Justify your answers.

Hint. How does an open ball with radius $0 < r < 1$ look like in $(\mathbb{R}^2, d^*)$?

An open ball with centre $p = (p_1, p_2)$ of radius $0 < r < 1$ consists of the points $q = (q_1, q_2)$ such that $|q_1 - p_1| < r$ and $q_2 = p_2$.

(a) $[-1,1] \times [-1,1]$,
not open

(b) $(-1,1) \times [-1,1]$,
open

(c) $\{p : \|p\| = p_1^2 + p_2^2 < 1\}$,
open

(d) $\{p : \|p\| = p_1^2 + p_2^2 \leq 1\}$.
not open