Week 5 Extreme Value Theory

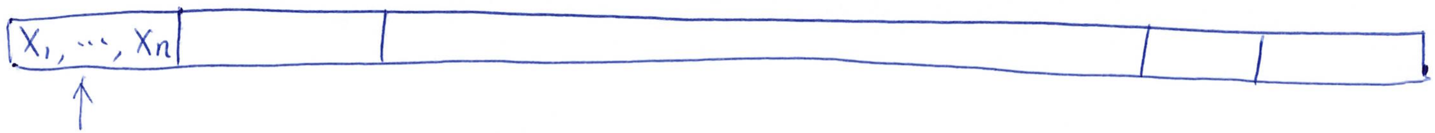
Two approaches : ① ~~genear~~ generalised extreme value distributions

GEV

② Generalised Pareto distributions

GPD

1. GEV



Block of

Size n

$X_1, \cdots, X_n$ IID

$X_M = \max\{X_1, X_2, \cdots, X_n\}$ the block maxima

$$P(X_M \le x) = P(X_1 \le x,\ X_2 \le x,\ X_3 \le x,\ \dots,\ X_n \le x)$$

$$\overset{\text{independent}}{=} P(X_1 \le x) \cdot P(X_2 \le x) \cdots P(X_n \le x)$$

$$\overset{\text{identical}}{=} [\underbrace{P(X \le x)}]^n$$

$$P(X_M \le x) = [F(x)]^n \quad (*)$$

$\alpha_1, \alpha_2, \dots, \alpha_n$
$\beta_1, \beta_2, \dots, \beta_n$ } suitable sequences of real constants

We standardise the values of X_M

the standardised block maxima : $\dfrac{X_M - \alpha_n}{\beta_n}$

$$\lim_{n\to\infty} P\left(\frac{X_M - \alpha_n}{\beta_n} \le x\right) \overset{*}{=} \lim_{n\to\infty} [F(\beta_n x + \alpha_n)]^n$$

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e.g. α mean, β std $\quad \dfrac{X_M - \alpha_n}{\beta_n} \to N(0,1)$ when $n \to \infty$

EVT : it is possible to find such values of α_n, β_n for most X

$n \uparrow$ $\quad \dfrac{X_M - \alpha_n}{\beta_n} \xrightarrow{\text{converge}}$ GEV distribution with CDF:

↑ standardised block max

$$\lim_{n \to \infty} \left[F(\beta_n x + \alpha_n) \right]^n = H(x)$$

$$H(x) = \begin{cases} \exp\left\{ -\left(1 + \dfrac{\gamma(\cancel{\alpha} x - \alpha)}{\beta}\right)^{-\frac{1}{\gamma}} \right\} & \gamma \neq 0 \\ \exp\left\{ -\exp\left(- \dfrac{(x-\alpha)}{\beta} \right) \right\} & \gamma = 0 \end{cases}$$

Generalized Pareto Distribution GPD

Def extreme value $X > u$ $\quad -X < -u$

E.g. XOL Reinsurance

$-\ln L$ min

$u \longrightarrow M$

$\ln L$ max

Let X be a r.v. CDF $F_X(x)$

<u>Excess</u> over the threshold u $\quad X-u \mid \underbrace{X>u}_{\text{extreme event}}$

$x_F \overset{<}{\cancel{=}} \infty$

$$P(A|B) = \frac{P(A \& B)}{P(B)}$$

CDF of the excess $\quad (0 \le x \le x_F - u)$

$$F_u(x) = P(X-u \le x \mid X > u) \underset{\text{of con Prob}}{\overset{\text{def}}{=}} \frac{P(X-u \le x \ \& \ X > u)}{P(X > u)}$$

$$= \frac{P(X \le x+u \,\&\, X > u)}{P(X > u)}$$

$$= \frac{P(X \le x+u) - P(X \le u)}{P(X > u)}$$

$$= \frac{F(x+u) - F(u)}{1 - F(u)}$$

. . . . . .

The distribution of the threshold exceedances will converge to a GPD as $u \uparrow$

i.e. $\lim_{u \to \infty} F_u(x) = G(x)$

PDF of $G(x)$:

$$G(x) = \begin{cases} 1 - \left(1 + \frac{x}{\gamma \beta}\right)^{-\gamma} & \gamma \neq 0 \\ 1 - \exp\left(-\frac{x}{\beta}\right) & \gamma = 0 \end{cases}$$

γ shape

β: scale

Measures of tail weight

4 ways to measure tail weight

1. The existence of moments

Review: $E(X^k)$ non-central moments

$$E(X^k) = \int_0^\infty x^k f(x)\,dx$$ ~~conf~~ continuous X

E.g. ① Gamma (α, λ)

PDF $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$

Fact: the kth moment exists for all values of k

less constraints

② Pareto (α', λ')

PDF: $f(x) = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}$

Fact: The kth moment only exists for $\alpha < \lambda$

Conclusion: Gamma lighter tail than Pareto

2. Limiting density ratios

limiting density ratio (LDR) $= \boxed{\lim_{x\to\infty} \frac{f_{X_1}(x)}{f_{X_2}(x)}}$

$f_{X_1}(x) \to$ PDF X_1

$f_{X_2}(x) \to$ PDF X_2

$\begin{cases} LDR \to 0 & \Rightarrow X_1 \text{ lighter tail} \\ LDR \to \infty & \Rightarrow X_2 \text{ lighter tail} \end{cases}$

e.g. $\frac{\text{PDF Gamma}}{\text{PDF Perato}}$ $x \to \infty$ $e^{-\lambda x} \to 0$ $\Rightarrow$ Gamma has lighter tail

3. Hazard rate

hazard rate $h(x) = \frac{f(x)}{\boxed{1-F(x)}}$

$h(x) \to$ force of mortality

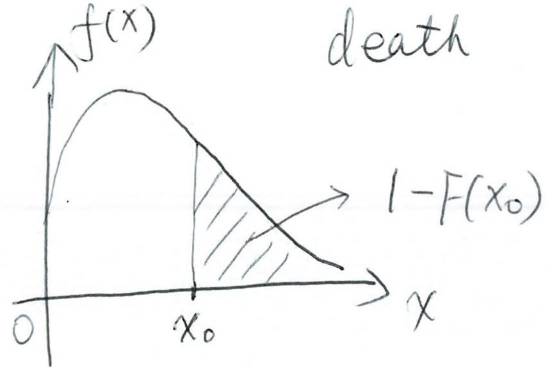
$f(x) \to$ PDF

$f(x) \to$ rate of failure

$1-F(x) \to$ survival up until that point

$F(x) \to$ CDF

e.g. Pareto



$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\dfrac{\alpha\lambda^{\alpha}}{(\lambda+x)^{\alpha+1}}}{\left(\dfrac{\lambda}{\lambda+x}\right)^{\alpha}}$$

$$= \frac{\alpha}{\lambda+x}$$

$h(x)$ decreasing func of x $\Rightarrow$ heavy tail

4. Mean residual life $\rightarrow e(x)$

$$e(x) = \frac{\int_x^{\infty}(y-x)f(y)\,dy}{\int_x^{\infty}f(y)\,dy} = \frac{\int_x^{\infty}\{1-F(y)\}\,dy}{1-F(x)}$$

$\rightarrow$ the expected remaining survival time.

$\rightarrow$ given survival up until the point.

$\downarrow$

expected future lifetime

e.g. Pareto

$$e(x) = \frac{\int_x^\infty \{1-F(y)\}\,dy}{1-F(x)} = \frac{\int_x^\infty \left(\frac{\lambda}{\lambda+y}\right)^\alpha dy}{\left(\frac{\lambda}{\lambda+x}\right)^\alpha}$$

$$= (\lambda+x)^\alpha \int_x^\infty (\lambda+y)^{-\alpha}\,dy$$

$$= \frac{\lambda+x}{\alpha-1}$$

$e(x)$: increasing function of x ⇒ heavy tail