MTH6127 **Metric Spaces and Topology**

Course work 3 **9 February 2024**

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1. Let (X, d) be a metric space, and let $\sigma : X \times X \to \mathbb{R}$ be the metric on X defined by $\sigma(x, y) = \min\{d(x, y), 1\}$. Compare the open balls $B^d(c; r)$ and $B^\sigma(c; r)$ with respect to the metrics d and σ.

   The balls $B^d(x; r)$ and $B^\sigma(x; r)$ coincide for $r \leq 1$. For $r > 1$ the ball $B^\sigma(x; r)$ equals X.

2. Show that a subset $U \subset X$ is open with respect to the metric d if and only if it is open with respect to σ (defined above).

   Open sets are unions of open balls of radius $r \leq 1$. Since the open balls of radius $r \leq 1$ for d and for σ coincide, the families of open sets for d and for σ are also identical.

3. Let (X, d) be a metric space such that for all $x, y \in X$ with $x \neq y$ one has $d(x, y) \geq 1$. Describe in this metric the open balls and open and closed sets.

   The open ball $B(c; r)$ of radius $r < 1$ equals $\{c\}$ - the single point. In this case any subset $U \subset X$ is open and closed.

4. Show that a subset $U \subset \mathbb{R}^m$ is open (closed) with respect to the metric d_p, where $p \in [1, \infty]$, if and only if it is open (closed) with respect to the metric d_∞.

   We shall use the inequalities

   $$d_\infty(v, w) \leq d_p(v, w) \leq m^{\frac{1}{p}} \cdot d_\infty(v, w), \quad v, w \in \mathbb{R}^m.$$

   Thus,

   $$B^{d_p}(c; r) \subset B^{d_\infty}(c; r) \subset B^{d_p}(c;\, m^{1/p} \cdot r).$$

   We want to show that any d_∞-open ball is d_p-open and any d_p-open ball is d_∞-open. Indeed, if $B = B^{d_\infty}(c; r)$ then

   $$\begin{aligned} B &= \bigcup_{c' \in B} B^{d_\infty}(c'; r - d_\infty(c, c')) \\ &= \bigcup_{c' \in B} B^{d_p}(c'; r - d_\infty(c, c')) \end{aligned}$$

is d_p-open. Similarly, if $B' = B^{d_p}(c;r)$ then

$$\begin{aligned} B' &= \bigcup_{c' \in B'} B^{d_p}(c'; r - d_p(c,c')) \\ &= \bigcup_{c' \in B'} B^{d_\infty}(c'; m^{-1/p}(r - d_p(c,c'))) \end{aligned}$$

and we see that B' is d_∞-open.

5. For any $p > 0$ we may define a function $||\cdot||_p : \mathbb{R}^m \to \mathbb{R}$ by the usual formula

$$||v||_p = \left[\sum_{i=1}^{m} |x_i|^p \right]^{1/p}, \quad v = (x_1, x_2, \ldots, x_m)$$

Show that this function does not satisfy the triangle inequality for $p \in (0,1)$ if $m > 1$.

Let $e_i \in \mathbb{R}^m$ denote the vector having only one nonzero coordinate 1 on position i. Then $||e_i||_p = 1$ and $||e_i + e_j||_p = 2^{1/p}$. We see that the inequality $||e_i + e_j||_p \leq ||e_i||_p + ||e_j||_p$ is satisfied only for $p \geq 1$.

6. Let $d(x,y) = |x - y|$ be the standard metric on the real line $\mathbb{R}$ and let $\sigma(x,y) = \min\{d(x,y), 1\}$ as above. Consider the set $\mathbb{R}^\omega$ of all infinite sequences $\mathbf{x} = (x_n)$ of real numbers $x_n \in \mathbb{R}$, where $n = 1, 2, \ldots$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\omega$, where $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, define

$$D(\mathbf{x}, \mathbf{y}) = \sup_{n \geq 1} \left\{ \frac{\sigma(x_n, y_n)}{n} \right\}.$$

Show that D is well-defined and is a metric on $\mathbb{R}^\omega$.

Clearly, $\sigma(x_n, y_n) \leq 1$ and $\frac{\sigma(x_n,y_n)}{n} \leq 1/n$; therefore the supremum above is finite and $D(\mathbf{x}, \mathbf{y}) \in [0,1]$. (M1) and (M2) are obvious. To show (M3) we note that $\sigma(x_n, y_n) \leq \sigma(x_n, z_n) + \sigma(z_n, y_n)$, and $\frac{\sigma(x_n,y_n)}{n} \leq \frac{\sigma(x_n,z_n)}{n} + \frac{\sigma(z_n,y_n)}{n}$, and taking the supremum of both sides of the last inequality, $D(\mathbf{x}, \mathbf{y}) \leq D(\mathbf{x}, \mathbf{z}) + D(\mathbf{z}, \mathbf{y})$.