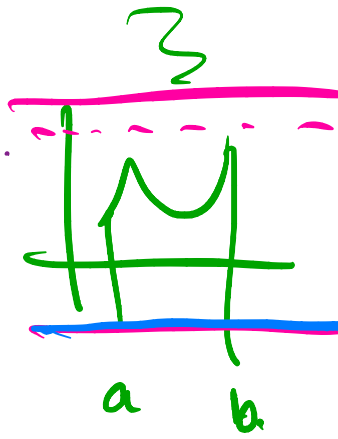
§6. Riemann Integration & Uniform Convergence

WEEK 5

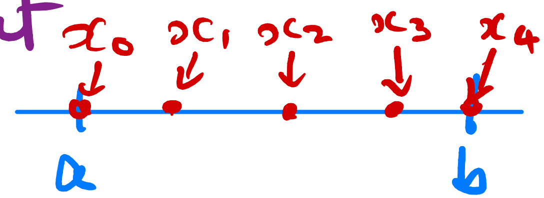
§6.1 Definition of Riemann Integral

Let us assume $f : [a,b] \to \mathbb{R}$ is bounded.



Def^n 6.1.1 (Partition)

A partition P of the interval $[a,b]$ is a sequence of real numbers $P = \{x_i\}_0^n$ such that



$$a = x_0 < x_1 < \dots < x_n = b.$$

Denote set of all partitions on the interval $[a,b]$ by $\mathcal{P}$.

e.g. $P = \{a = x_0, x_1, \dots, x_n = b\} \in \mathcal{P}$. $\quad \mathcal{P}_{[a,b]}$

Let $\Delta x_i = x_i - x_{i-1}$, $i = 1$ to n

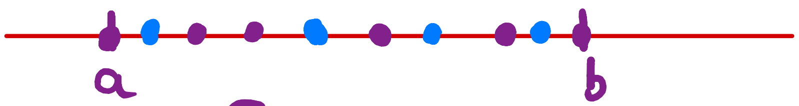
P creates n-intervals $\Delta x_1, \dots, \Delta x_n$ in $[a,b]$.

The **mesh** of P, denoted $\sigma(P) = \max\{\Delta x_i \mid i=1,\dots,n\}$.

The partition P is said to be "**equidistant**" if

$$\Delta x_i = \frac{b-a}{n}, \quad \forall i=1,\dots,n.$$

A partition P' is said to be a **refinement** of P if the set of partition points of P is a subset of those of P'.

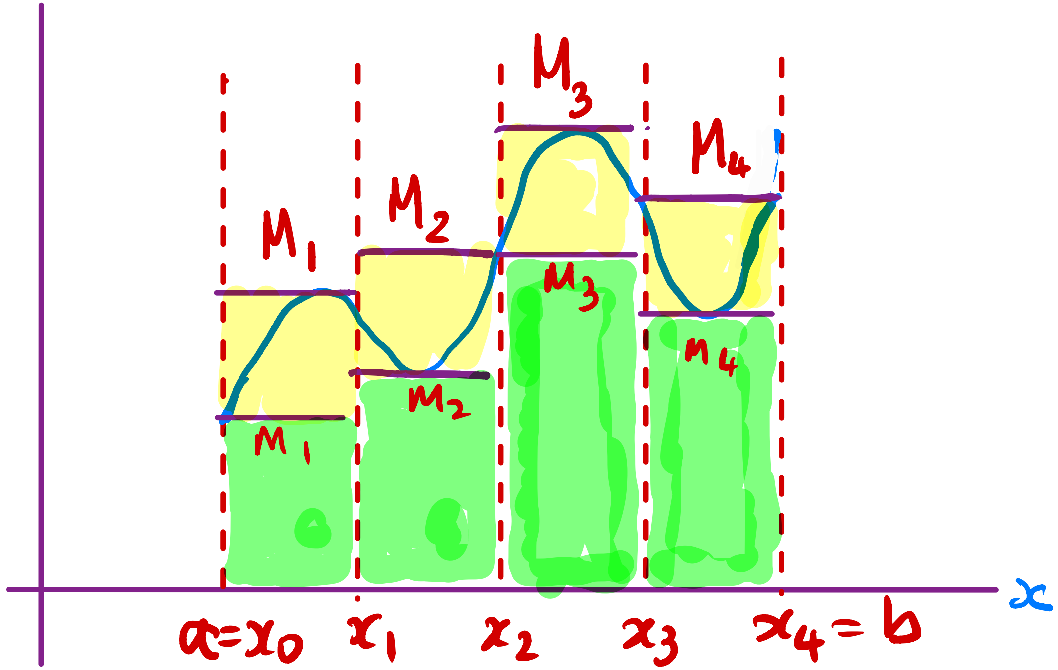


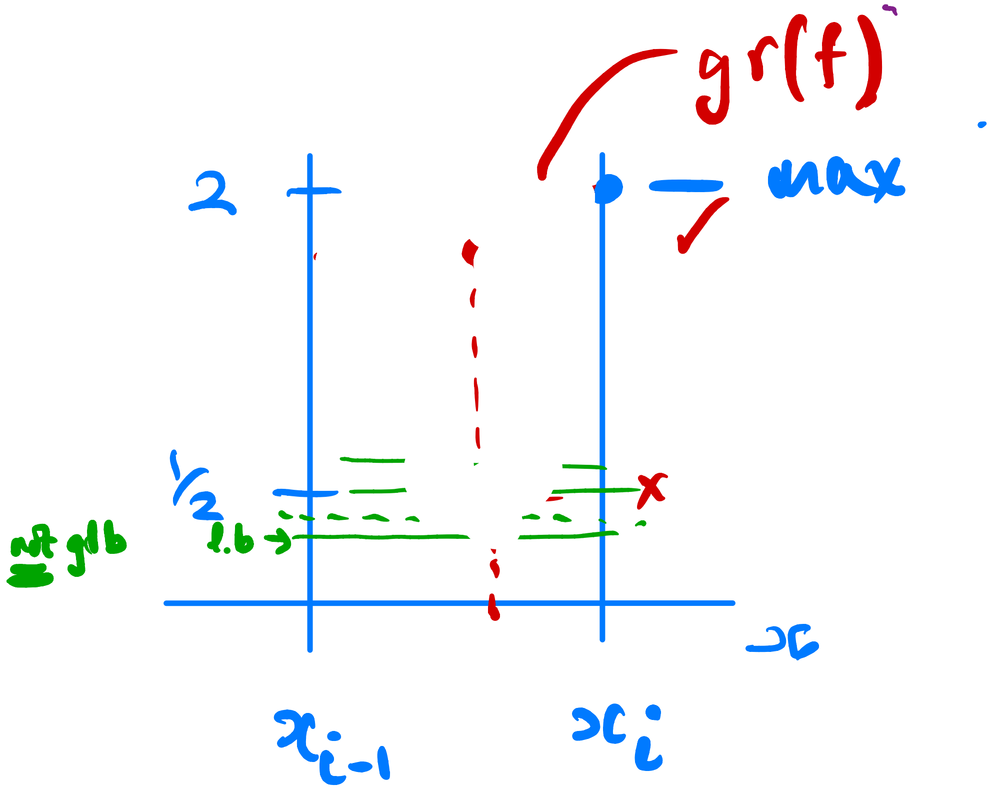
$\{\bullet\} = P \qquad \{\bullet\ \bullet\} = P'$

Example $\quad P = \{0, 1, 2, 3\}$ on the interval $[\underset{a}{0}, \underset{b}{3}]$

$P' = \{0, 0.5, 1, 2, 2.5, 2.75, 3\}$ on $[\underset{a}{0}, \underset{b}{3}]$

then $P' \geq P$, P' is a refinement of P.





Lower approximative sum $= \sum_{i=1}^{n} m_i \Delta x_i \quad , m_i = \inf_{[x_{i-1}, x_i]} f(x)$

Upper approximative sum $= \sum_{i=1}^{n} M_i \Delta x_i \quad , M_i = \sup_{[x_{i-1}, x_i]} f(x)$

Defⁿ 6.1.2 Let $f:[a,b]\to\mathbb{R}$ be bounded and let $P=\{x_0,\dots,x_n\}$ be a partition of $[a,b]$

We define: **Lower sum** of f w.r.t. P as

$$L(f,P)=\sum_{i=1}^{n}\inf_{[x_{i-1},x_i]} f(x)\,\Delta x_i=\sum_i m_i\Delta x_i$$

and **Upper sum** of f w.r.t. P as

$$U(f,P)=\sum_{i=1}^{n}\sup_{[x_{i-1},x_i]} f(x)\,\Delta x_i=\sum_i M_i\Delta x_i$$

$$m_i=\inf_{[x_{i-1},x_i]} f(x)$$

$$M_i=\sup_{[x_{i-1},x_i]} f(x)$$

If $|f(x)|\le M$ for all $x\in[a,b]$, then we have

$$-M(b-a)\le L(f,P)\le U(f,P)\le M(b-a)$$

Why?

$$-M \le f(x) \le M$$

$-M$ is a l.b. for f on $[a,b]$

$\therefore -M$ is a l.b. for f on $[x_{i-1}, x_i] \subseteq [a,b]$

$$\therefore -M \le \underset{[x_{i-1},x_i]}{\overset{\text{inf}}{\text{glb}}} f(x) \le \underset{[x_{i-1},x_i]}{\overset{\text{sup}}{\text{lub}}} f(x) \le M$$

$-M \le m_i \le M_i \le M$

$$-M\Delta x_i \le m_i\Delta x_i \le M_i\Delta x_i \le M\Delta x_i$$

$$\therefore \sum_{i=1}^{n} \text{''} \le \sum_{i=1}^{n} \text{''} \le \sum_{i=1}^{n} \text{''} \le \sum_{i=1}^{n} \text{''}$$

Note $\sum \text{''} = \sum_{i=1}^{n} M_i\Delta x_i$

$$-M(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$$

Lemma 6.1.3 If P' is a refinement of P on the interval $[a,b]$

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P)$$

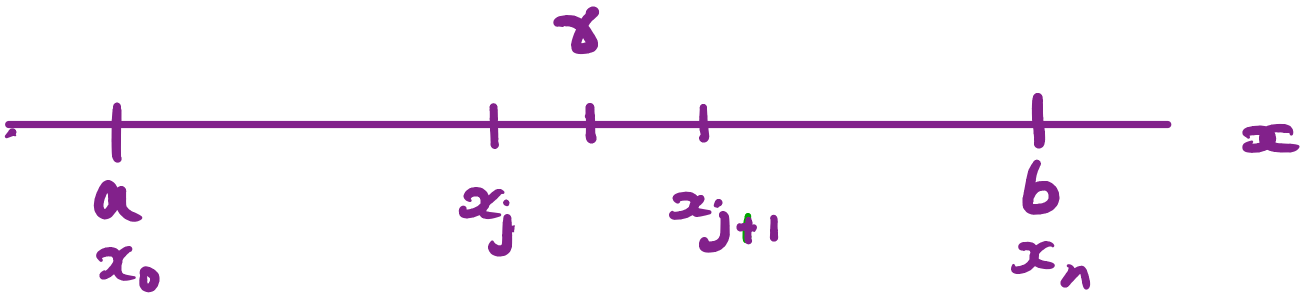
Proof

Let $P = \{x_0, x_1, \ldots, x_n\}$

Consider $P' = \{x_0, x_1, \ldots, x_j, \gamma, x_{j+1}, \ldots x_n\}$, i.e. $x_j < \gamma < x_{j+1}$.

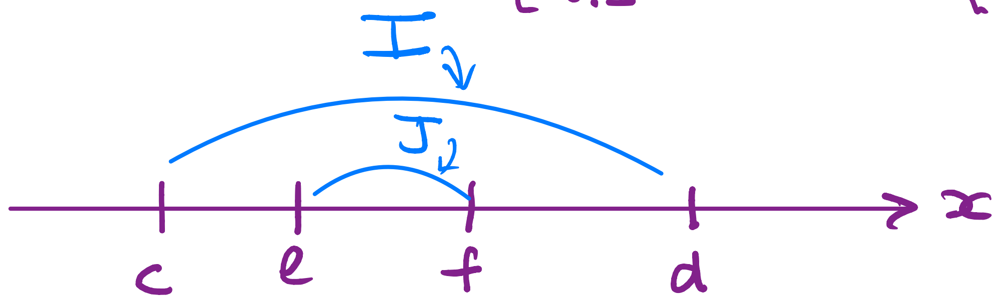
Note $x_0 = a$ & $x_n = b$ remain fixed.

One extra value is added in the partition at $x = \gamma$ between x_j and x_{j+1}



In the upper and lower sums, $\inf f(x)$ and $\sup f(x)$ are required!
[??] [??]

Consider the following



with $f(x)$ defined on $[c,d]$. Let $I = [c,d]$, $J = [e,f]$

Note $J \subseteq I$. Let M_I be $\underset{x\in I}{\text{l.u.b.}}\{f(x)\} \Rightarrow M_I$ is upper bound of f on I

$\Rightarrow M_I$ is upper bound for f on $J \Rightarrow M_J \le M_I$ (M_J is the least U.B)

Similarly $m_I \le m_J$ (greatest lower bounds!)

Then $L(f,P') - L(f,P) = \geq 0$

$$= \inf_{[x_j,\gamma]} f(x)\,[\gamma - x_j] + \inf_{[\gamma,x_{j+1}]} f(x)\,[x_{j+1} - \gamma]$$

$$- \inf_{[x_j,x_{j+1}]} f(x)\,(x_{j+1} - x_j)$$

P' terms
2 rectangles
$[x_j, \gamma]$, $[\gamma, x_{j+1}]$.

$$\geq \inf_{[x_j,x_{j+1}]} f(x) \left\{ [\gamma - x_j] + [x_{j+1} - \gamma] - [x_{j+1} - x_j] \right\}$$

$$= 0$$

Note $\inf\limits_{x \in I} f(x) \leq \inf\limits_{x \in J} f(x)$ if $J \subseteq I$

Similarly, for $U(f,P)$. Let $P' = \{x_0, x_1, \ldots, x_{j-1}, \gamma, x_j, \ldots, x\}$.

Then P' is a refinement of P (an extra vertex has been added!!)

$$\boxed{U(f,P')} = \sum_{i=1}^{n} \Big(\sup_{[x_{i-1},x_i]} f\Big)(x_i - x_{i-1})$$

$$= \sum_{\substack{i=1 \\ i\neq j}}^{n} \sup_{[x_{i-1},x_i]} f \,(x_i - x_{i-1}) + \sup_{[x_{j-1},\gamma]} f \,(\gamma - x_{j-1}) + \sup_{[\gamma,x_j]} f\,(x_j - \gamma)$$

$$\boxed{\leq} \quad '' \quad + \sup_{[x_{j-1},x_j]} f\,(\gamma - x_{j-1}) + \sup_{[x_{j-1},x_j]} f\,(x_j - \gamma)$$

$$= \quad '' \quad + \sup_{[x_{j-1},x_j]} f \Big[(\gamma - x_{j-1}) + (x_j - \gamma)\Big]$$

$$(x_j - x_{j-1})$$

$$\boxed{= U(f,P)}$$

SIMILAR ARGUMENT for $L(f,P)$ gives $L(f,P) \leq L(f,P')$ WHY?

So we get P' is a refinement of P and

$$\boxed{L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P)}$$

<u>**Def^n 6.1.6**</u> (Upper integral & Lower Integral)

The number given by

$$\underline{\int_a^b} f\,dx = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$$

is called the lower integral of f, and

$$\overline{\int_a^b} f\,dx = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$$

is called the upper integral of f.

<u>Lemma 6.1.7</u> If P_1 and P_2 are partitions of $[a,b]$, then

$$L(f,P_1) \leq U(f,P_2)$$

Let $P = P_1 \cup P_2$ (the points of both P_1 & P_2). Then P is a refinement of both P_1 & P_2. Therefore

$$L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2)$$

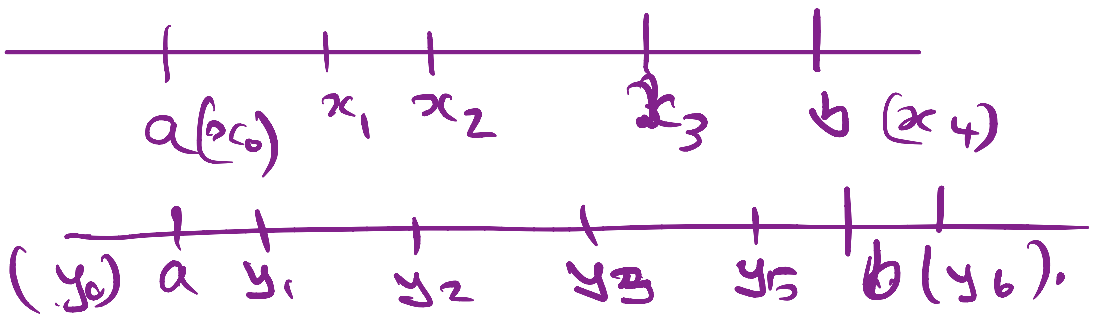
refinement refinement.

From this we get

$$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx$$

$P_1 \cup P_2$

- common refinement.



P_1. Lower

P_2' Upper.

Def^n 6.1.8 (Riemann Integrable)

if $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$, then f is integrable

& $\int_a^b f\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

L(f,P)'s   U(f,P)'s

is the integral of f over [a,b].

Note Given $P_1, P_2 \in \mathcal{P}$: $L(f,P_1) \leq U(f,P_2)$

For any $Q \in \mathcal{P}$

$$L(f,P) \leq U(f,Q), \text{ for all } P \in \mathcal{P}$$

$\therefore$ $U(f,Q)$ is an upper bound of the set $\{L(f,P) \mid P \in \mathcal{P}\}$

$$\Rightarrow \underline{\int_a^b} f(x)\,dx = \operatorname*{lub}_{P \in \mathcal{P}} L(f,P) \leq U(f,Q), \quad \forall Q \in \mathcal{P}$$

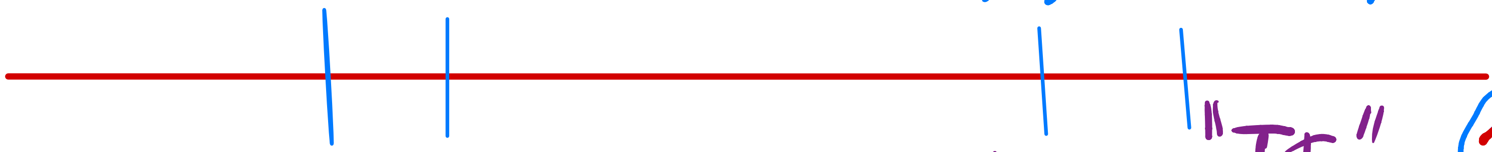
Therefore $\underline{\int_a^b} f(x).dx$ is a lower bound for all $U(f,Q)$

$$\Rightarrow \underline{\int_a^b} f(x).dx \leq \operatorname*{glb}_{Q \in \mathcal{P}} \{U(f,Q)\} = \overline{\int_a^b} f(x)\,dx$$

Picture view of upper and lower sums

Let P, Q be partitions. Let $R = P \cup Q$

$L(f,P) < L(f,R) < U(f,R) < U(f,Q)$

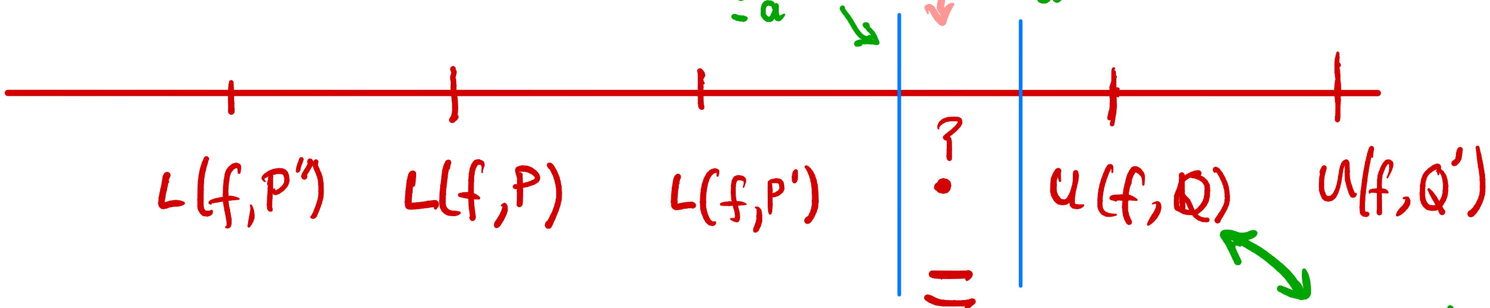


$\therefore$ Any $L(f,P) \le$ Any $U(f,Q)$

"IF" $\boxed{=}$ integral exists.

$\int_a^b f\,dx$

$\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$ "upper integral"



$\text{lub}_P \{L(f,P)\}$ $\quad$ $\text{glb}_Q \{U(f,P)\}$

upper bound of the $L(f,P)$'s

RIEMANN INTEGRABILITY CONDITION

Thm 6.1.9 The function $f:[a,b]\to\mathbb{R}$ is integrable if and only if for every $\varepsilon>0$, there exists a partition P such that

$$U(f,P)-L(f,P)<\varepsilon$$

Proof

$\Leftarrow$ Given $\varepsilon>0$, $\exists$ $P\in\mathcal{P}$ such that

such that $U(f,P)-L(f,P)<\varepsilon$

and $U(f,P)\geq \overline{\int_a^b} f\,dx \geq \underline{\int_a^b} f\,dx \geq L(f,P)$

$$\Rightarrow\quad 0\leq \overline{\int_a^b} f\,dx-\underline{\int_a^b} f\,dx<\varepsilon \Rightarrow$$

$$\Rightarrow \overline{\int_a^b} f\,dx-\underline{\int_a^b} f\,dx=0 \quad\Rightarrow\quad \int_a^b f\,dx \text{ exists.}$$

2 $$\int_a^b f\,dx = \underline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx$$

is the integral of f over $[a,b]$.

$\Rightarrow$ if $\int_a^b f\,dx$ exists, then $\underbrace{\underline{\int}_a^b f\,dx}_{\text{lower integral}} = \underbrace{\overline{\int}_a^b f(dx)}_{\text{upper integral}}$

Given $\varepsilon > 0$, $\exists$ $P_1, P_2 \in \mathcal{P}$

$A = \int_a^b f\,dx - L(f,P_1) < \varepsilon/2$ and $B = U(f,P_2) - \int_a^b f\,dx < \frac{\varepsilon}{2}$

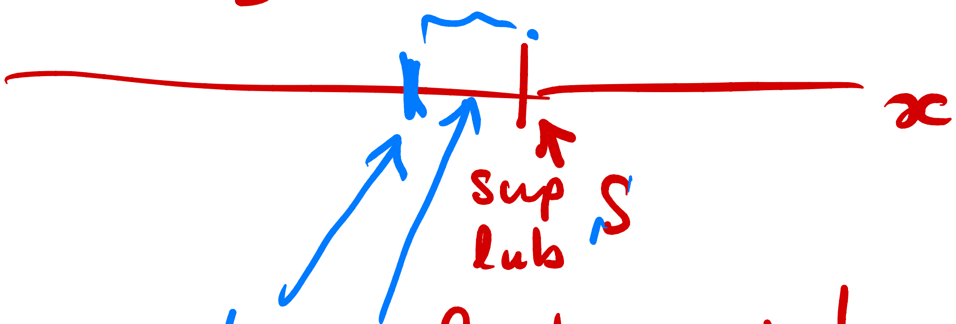
$$\Rightarrow U(f,P_2) - L(f,P_1) = A + B < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Let $P = P_1 \cup P_2$, then

$$U(f,P) - L(f,P) < U(f,P_2) - L(f,P_1) < \varepsilon.$$

lub

$S \subseteq \mathbb{R}$



least upper bound

$\exists s \in S$ such that $t < s \le \sup S$

let $l = \text{lub}\,\{s\}_{s \in S}$

if $l' < l$, l' is <u>not</u> an upper bound

$\therefore \exists s \in S$ such that $l' < s \le l$

let $g = \text{glb}\, S$

if $g' > g$, g' is not a lower bound

$\therefore \exists s \in S$ such that $g \le s < g$

Example 6.1.10 Define the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \ x \in [0,1] \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q}, \ x \in [0,1] \end{cases}$$

Taking any partition $P = \{x_0, x_1, \dots, x_n\}$,

then $L(f,P) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x) \ (x_i - x_{i-1})$

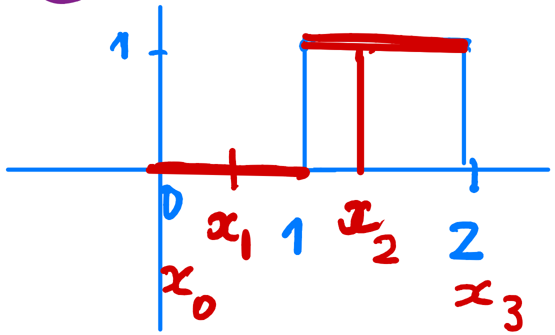
In any non-trivial closed interval ($x_{i-1} \neq x_i$) $\exists$ irrational nos (how?) $\therefore$ $\inf_{[x_{i-1}, x_i]} f(x) = 0$ & $L(f,P) = 0$

Similarly $\sup_{[x_{i-1}, x_i]} f(x) = 1$ & $U(f,P) = 1$ for any $P \in \mathcal{P}$ $\therefore$ Integral does not exist

Note $\underline{\int_a^b} f(x)\,dx = 0 \ {=L(f,P)} \ \forall P$ $\quad \overline{\int_a^b} f(x)\,dx = 1 \ {=U(f,P)} \ \forall P$

Not equal

(ii) Let $f:[0,2]\to\mathbb{R}$ be defined by



$$f(x)=0 \qquad 0\le x<1$$
$$f(x)=1 \qquad 1\le x\le 2$$

Choose a partition P on $[0,2]$ such that

$$0=x_0<x_1<x_2=1<x_3=2$$

$$\sup_{[x_0,x_1]} f(x)=0,\quad \inf_{[x_0,x_1]} f(x)=0,\quad \sup_{[x_1,x_2]} f(x)=1,\quad \inf_{[x_1,x_2]} f(x)=0$$

$$\sup_{[x_1,x_2]} f(x)=1,\quad \inf_{[x_2,x_3]} f(x)=1$$

What are $U(f,P)$ in this example?
$L(f,P)$

$\therefore \quad U(f,P) = 0(x_1-x_0) + 1(x_2-x_1) + 1(x_3-x_2) = x_3-x_1$

$L(f,P) = 0(x_1-x_0) + 0(x_2-x_1) + 1(x_3-x_2) = x_3-x_2$

$\therefore \quad U(f,P) - L(f,P) = (x_3-x_1) - (x_3-x_2) = (x_2-x_1)$

Given $\varepsilon > 0$ $U(f,P) - L(f,P) < \varepsilon$ provided $x_2 - x_1 < \varepsilon$

Note that $U(f,P) - L(f,P) = 0$ provided $x_2 - x_1 = 0$

It follows that f is integrable by choosing $x_2 - x_1 < \varepsilon$

In fact for this function f the integral is given by

$U(f,P) = L(f,P)$ when $x_2 - x_1 = 0$ and $P = \{0,1,2\}$.

This situation is special!. Typically for a given partition $U(f,P) \neq L(f,P)$! Note: $x_2 = x_1 \Rightarrow gr(f)$ "matches" P

END OF WEEK 5