Mathematical Tools for Asset Management MTH6113

Mean Variance Portfolio Theory

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Plan

- The Scope of Mean Variance Portfolio Theory
- Example
- Diversification
- Mean Variance Frontier and Efficient Frontier
- Mean Variance Portfolio Theory
- Choosing an Optimal Efficient Portfolio

Mean Variance Portfolio Theory

MVPT - modern portfolio theory where investors choose optimal portfolios based on risk and return:

- maximising the return for a specified risk, or
- minimising the risk for a specified return

Mean Variance Portfolio Theory

MVPT vs Utility Theory?

- *MVPT* leads to optimal portfolio choices consistent with the *maximisation of expected utility* if either:
  - investors are assumed to have quadratic utility functions or
  - investment returns are assumed to be normally or lognormally distributed

The decision is based on **mean** and **variance of investment returns** over a single time horizon

Example

Assume that there are only three possible states of the economy (recession state, normal state and an expansion state). From past experience and your personal beliefs, you expect the economy will be in a recession state 25% of the time, in the normal state 50% of the time and in the expansion state 25% of the time. There are two stocks, A and B with the following returns in different states of the economy.

	Probability	Ret A	Ret B
Recession	0.25	-4%	0%
Normal	0.5	4%	1%
Expansion	0.25	5%	10%

Example

The summary statistics of the example (please see Problem Set 1):

	Return A	Return B
Expected Return	2.25%	3.00%
Variance	0.001319	0.001650
St. Dev	3.6315%	4.0620%
Covariance	0.000775	
Correlation	0.525386	

- What are the expected return and variance on a portfolio of 50% stock A and 50% stock B?
- What are the expected return and variance on a portfolio of 70% stock A and 30% stock B?
- What are the expected return and variance on a portfolio of 30% stock A and 70% stock B?

Example

Portfolio	50% - 50%	70% - 30%	30% - 70%
Expected Return	2.63%	2.48%	2.78%
Variance	0.001130	0.001120	0.001253
St. Dev	3.3611%	3.3469%	3.5393%

- Questions:
- standard deviation: of portfolios vs stock A and stock B individually.
- stock B higher expected return but also higher risk (standard deviation).
- benefit of holding any stock in the portfolio
  - it reduces the standard deviation of the portfolio.

Diversification

Reminder:

The expected return on a portfolio of N assets with returns R_i and weights w_i is:

$$E\left(R_P\right) = \sum_{i=1}^{N} w_i E_i$$

where E_i is expected return of security/asset i.

Diversification

The variance of a portfolio of N assets with returns R_i and weights w_i is:

$$Var\left(R_P\right) = \sum_{i=1}^{N} w_i^2 V_i + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} w_i w_j C_{ij}$$

where V_i is the variance of return of security/asset i and C_{ij} the covariance between asset i and j

Diversification

Consider a portfolio P with:

- weight of each stock $\frac{1}{N}$
- individual variance of stock i: V_i
- covariance between returns on asset i and j: C_{ij}

Diversification

Variance of portfolio P:

$$
\begin{aligned}
Var\left(R_P\right) &= \sum_{i=1}^{N}\left(\frac{1}{N}\right)^2 V_i + \sum_{i=1}^{N}\sum_{i\neq j}^{N}\frac{1}{N}\frac{1}{N}C_{ij} = \\
&= \frac{1}{N}\sum_{i=1}^{N}\frac{1}{N}V_i + \frac{N-1}{N}\sum_{i=1}^{N}\sum_{i\neq j}^{N}\frac{1}{\left(N-1\right)N}C_{ij}
\end{aligned}
$$

- $\sum_{i=1}^{N}\frac{1}{N}V_i \equiv V^*$ the average variance in the portfolio
- $\sum_{i=1}^{N}\sum_{i\neq j}^{N}\frac{1}{(N-1)N}C_{ij} \equiv C^*$ the average covariance

Diversification

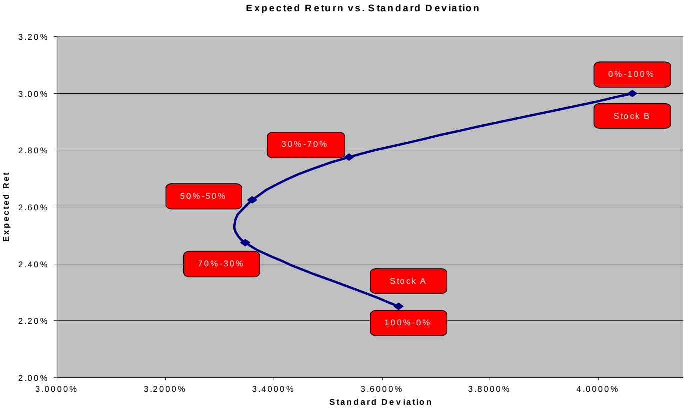
$$Var\left(R_P\right) = \frac{1}{N} V^* + \frac{N-1}{N} C^*$$

Result: As $N \rightarrow \infty$ $Var\left(R_P\right) \rightarrow C^*$

Diversification is one of the key concepts of modern finance

- The individual risk of the securities can be eliminated through diversification
- The contribution to the total risk caused by the covariance terms cannot be diversified away

Going back to the example



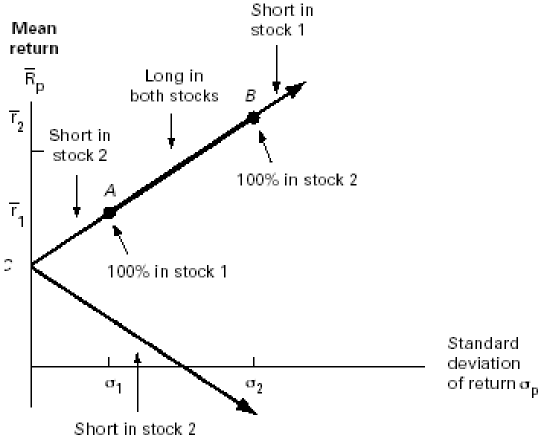
Going back to the example

If correlation between two stocks is less than 1 then st. dev.of portfolio less than weighted average st. dev. of its constituent assets

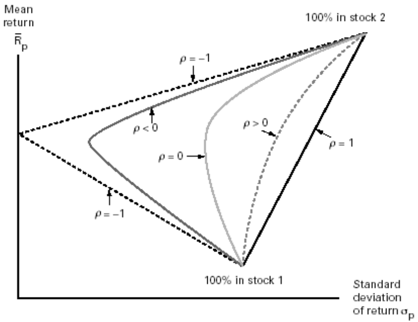
- *The investor can benefit from diversifying his portfolio*
- If the returns are perfectly negatively (positively) correlated a portfolio can be constructed with zero standard deviation
  - If assets positively correlated long one asset and short-sell the other
  - *short-selling*: selling something that you do not have (borrowed)
  - *taking a long position* in an asset: buying that asset

Diversification

Portfolios of Two Perfectly Positively Correlated Assets



Diversification



If correlation coefficients of –1 and +1 we can obtain risk-free portfolios with zero standard deviation of return

Mean Variance Portfolio Theory

Reminder: The aim is to find the optimal investment portfolio

Assumptions

1. all expected returns, variances & covariances of returns are known
2. investor decisions based on expected return and variance
3. investors non-satiated (prefer more to less)

Assumptions

4. investors risk averse
5. fixed single time horizon
6. no taxes or transaction costs
7. assets can be held in any amount

Mean Variance Portfolio Theory

Opportunity set specifies the options open to the investor.

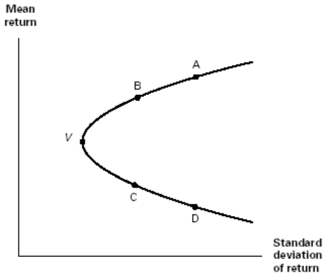
A portfolio is

- **efficient if investor cannot find a 'better' one**
- **inefficient if investor can find a 'better' one**
  - 'better' is in terms of - lower variance with same expected return or - higher expected return with same variance

The set of efficient portfolios is called **the efficient frontier**

Mean Variance Portfolio Theory

Mean-variance frontier is the expected return-variance locus:



Minimum variance portfolio: point V on the frontier

Optimal portfolio lies on the frontier and to the right of V

- investor's preferences are increasing in expected return and decreasing standard deviation

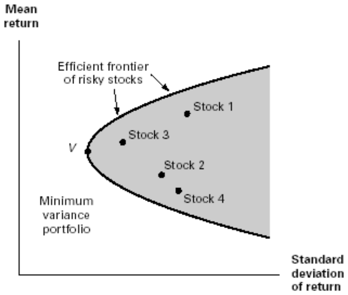
Important point: people could have different preferences between risk and return, so they might choose different locations on the frontier.

- B on the diagram represents and investor more risk averse than the investor located at A

Mean Variance Portfolio Theory

What happens if we have N assets in our portfolio?

- Analyze every single possible weight combination
- Graph the expected return of every portfolio as a function of st.dev.of portfolio



Mean Variance Portfolio Theory

N **assets case**

Mean-Variance Frontier the same shape as before

- the interior of the frontier consists of feasible but inefficient portfolios
- any portfolio or individual asset inside the frontier is dominated by the portfolios on the frontier

Mean Variance Portfolio Theory

***N* assets case**

Mean-Variance Frontier is the set of portfolios that:

- minimize risk for a given expected return or
- maximise expected return for a given risk

Mean Variance Portfolio Theory

Investment Decision is:

$$\min_{w_i} Var\left(R_P\right) \text{ such that}$$

$$\begin{aligned} E\left(R_P\right) &= E_P \\ \sum_{i=1}^{N} w_i &= 1 \end{aligned}$$

Constrained optimisation: use Lagrangian method!

Mean Variance Portfolio Theory

The Lagrangian function is:

$$\mathcal{L} = Var\left(R_P\right) + \lambda\left(E_P - \sum_{i=1}^{N} w_i E_i\right) + \mu\left(1 - \sum_{i=1}^{N} w_i\right)$$

or

$$\mathcal{L} = \sum_{i=1}^{N} w_i^2 V_i + \sum_{i=1}^{N}\sum_{i\neq j}^{N} w_i w_j C_{ij} + \lambda\left(E_P - \sum_{i=1}^{N} w_i E_i\right) + \mu\left(1 - \sum_{i=1}^{N} w_i\right)$$

First Order Conditions

At the optimum:

$$\frac{\partial \mathcal{L}}{\partial w_i} = 2\sum_{j=1}^{N} w_j C_{ij} - \lambda E_i - \mu^* = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \left(E_P - \sum_{i=1}^{N} w_i E_i \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = \left(1 - \sum_{i=1}^{N} w_i \right) = 0$$

System of $N+2$ equations with $N+2$ unknown. (not asked in the exam for SOC)

If we look at a portfolio of *two assets* only, using the constrains only allows us to find the opyimal choice.

For two security case **the global minimum variance** (point V on the diagram) occurs when:

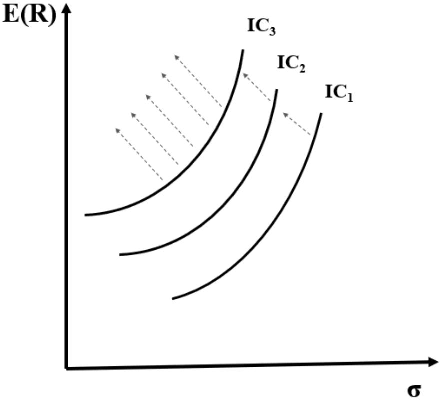
$$w_1 = \frac{V_2 - C_{12}}{V_1 - 2C_{12} + V_2}$$

(Please check! Optimisation with no restriction on the expected return)

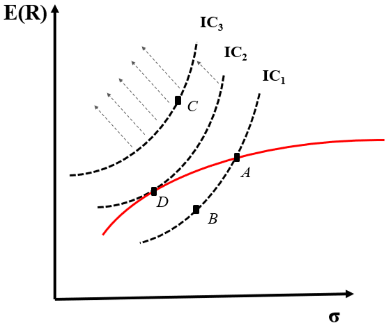
Choosing an Optimal Efficient Portfolio

- Recall - an *Indifference Curve* links all combinations of two bundles of goods which yield the same utility
- Portfolios on a single curve all give the same value of *EU* so the investor will be indifferent between them
- *Combining the investor's indifference curves with the efficient frontier of portfolios* allows us to determine *the investor's optimal portfolio*
- Expected utility is maximised by selecting the portfolio at the point where the **efficient frontier is at a tangent to an indifference curve**

Utility from risk and return



Choosing an Optimal Efficient Portfolio



- *A*,*B*, *D* feasable; *D*, *A* efficient, *D* optimal given investor's preference for risk!
- *D* is chosen by a more risked averse person than *A*
- Analysis without risk-free assets
- Harry Markowitz: Portfolio Selection The Journal of Finance, Vol 7, No.1 (Mar. 1952) pp77-91