WEEK 5

Last lecture:

- Vectors:
  - Directional derivatives along curves: $\frac{d}{d\lambda}$
  - Geometric objects intrinsic to M and independent of the choice of coordinates
  - In a coordinate basis: $V = V^a \partial_a$
  - Under coordinate transf. $x^{a'} = x^{a'}(x^b)$

$$V^{a'} = \frac{\partial x^{a'}}{\partial x^b} V^b$$

<u>Tensors</u>: 1-forms

Having defined vectors, we can now consider dual vectors (aka 1-forms). They live on the cotangent space T_p^*, which can be thought of the space of linear maps $\omega : T_p \to \mathbb{R}$

→ gradients of functions and they act on vectors:

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} \in \mathbb{R}$$

→ like vectors, one-forms are defined at a point

- A natural basis for the 1-forms are the gradients of the coordinates x^a : $\{dx^a\}$
- Recall, given some basis vectors $\hat{e}_{(a)}$ of T_p, we can construct a basis for T_p^* demanding $\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta^a{}_b$

Then for a coordinate basis we have

$$dx^a(\partial_b) = \frac{\partial x^a}{\partial x^b} = \delta^a{}_b$$

$\Rightarrow$ $\{dx^a\}$ form a basis of T_p^* (1-forms) and hence we can write an arbitrary 1-form as

$$\omega = \omega_a dx^a$$

- The transformation rules for 1-forms under changes of coordinates are

basis: $$dx^{a'} = \frac{\partial x^{a'}}{\partial x^b} dx^b$$

components: $$\omega_{a'} = \frac{\partial x^a}{\partial x^{a'}} \omega_a$$

Tensors of arbitrary rank (K, ℓ)

- A (K,ℓ) tensor is a multilinear map from K dual vectors and ℓ vectors to $\mathbb{R}$
- The components in a coordinate basis are found by acting on the basis one-forms and vectors:

$$T^{a_1 \dots a_K}{}_{b_1 \dots b_\ell} = T(dx^{a_1}, \dots, dx^{a_K}, \partial_{b_1}, \dots \partial_{b_\ell})$$

$$\Rightarrow \quad T = T^{a_1 \dots a_K}{}_{b_1 \dots b_\ell} \, \partial_{a_1} \otimes \dots \otimes \partial_{a_K} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_\ell}$$

- Transformation rule:

$$T^{a_1' \dots a_K'}{}_{b_1' \dots b_\ell'} = \frac{\partial x^{a_1'}}{\partial x^{a_1}} \dots \frac{\partial x^{a_K'}}{\partial x^{a_K}} \frac{\partial x^{b_1}}{\partial x^{b_1'}} \dots \frac{\partial x^{b_\ell}}{\partial x^{b_\ell'}} T^{a_1 \dots a_K}{}_{b_1 \dots b_\ell}$$

- Example: $S_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \qquad (x^1 = x, \; x^2 = y)$

$$\rightarrow \quad S = S_{ab} \, dx^a \otimes dx^b = dx \otimes dx + x^2 \, dy \otimes dy$$

Coordinate transf: $x' = \frac{2x}{y}, \quad y' = \frac{y}{2}$

Inverse: $x = x'y', \quad y = 2y'$

Components of S in the new coordinates:

$$S_{a'b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} S_{ab}$$

$$S_{x'x'} = \left(\frac{\partial x}{\partial x'}\right)^2 S_{xx} + \frac{\partial x}{\partial x'} \frac{\partial y}{\partial x'} \left(S_{xy} + S_{yx}\right) + \left(\frac{\partial y}{\partial x'}\right)^2 S_{yy}$$

$$= (y')^2 + 0 + 0 = (y')^2$$

$$S_{x'y'} = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial y'} S_{xx} + \frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} S_{xy} + \frac{\partial y}{\partial x'} \frac{\partial x}{\partial y'} S_{yx}$$
$$+ \frac{\partial y}{\partial x'} \frac{\partial y}{\partial y'} S_{yy}$$

$$= y'x' + 0 + 0 \quad = x'y'$$

$S_{y'x'} = S_{x'y'}$ since $S_{ab} = S_{ba}$

$$S_{y'y'} = \left(\frac{\partial x}{\partial y'}\right)^2 S_{xx} + 2 \frac{\partial x}{\partial y'} \frac{\partial y}{\partial y'} S_{xy} + \left(\frac{\partial y}{\partial y'}\right)^2 S_{yy}$$

$$= (x')^2 + 0 + 4x^2 = (x')^2 + 4(x'y')^2$$

$$\Rightarrow \quad S_{a'b'} = \begin{pmatrix} (y')^2 & x'y' \\ x'y' & (x')^2 + 4(x'y')^2 \end{pmatrix}$$

Alternative: consider the transformation of the differentials

$$dx = y'\,dx' + x'\,dy'$$

$$dy = 2\,dy'$$

$$\Rightarrow \quad S = dx \otimes dx + x^2\, dy \otimes dy$$

$$= (y'\, dx' + x'\, dy') \otimes (y'\, dx' + x'\, dy')$$
$$+ (x'y')^2 (2\, dy') \otimes (2\, dy')$$

$$= (y')^2\, dx' \otimes dx' + x'y' (dx' \otimes dy' + dy' \otimes dx')$$
$$+ [(x')^2 + 4(x'y')^2]\, dy' \otimes dy'$$

- Note: the gradient of a scalar (i.e., partial derivative) is a $(0,1)$-tensor $\to$ 1-form:

$$d\phi = \partial_a \phi\, dx^a$$

but the partial derivative of a tensor is NOT a tensor:

Consider the transformation of $\partial_a W_b$ where W_a is a $(0,1)$ tensor:

$$\partial_{a'} W_{b'} = \frac{\partial}{\partial x^{a'}} W_{b'} = \frac{\partial x^a}{\partial x^{a'}} \partial_a \left(\frac{\partial x^b}{\partial x^{b'}} W_b \right)$$

$$= \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} \partial_a W_b + \frac{\partial^2 x^b}{\partial x^{a'} \partial x^{b'}} W_b$$

$\Rightarrow$ NOT a tensor!!

The metric

- Denoted by g_{ab} (η_{ab} is reserved for the Minkowski metric)

- g_{ab} is a symmetric and non-degenerate (i.e. $\det g \neq 0$) $(0,2)$ tensor $\rightarrow$ we can define the inverse metric g^{ab} as

$$g^{ab} g_{bc} = \delta^a{}_c \qquad (g^{ab} \text{ is also symmetric})$$

- The metric is important because:
  - Allows to compute the length of curves
  - Provides a notion of past and future
    $\rightarrow$ causal structure
  - It represents the gravitational field in GR

- The line element, i.e, infinitesimal distance is now given by:

$$ds^2 = g_{ab}(x)\, dx^a dx^b$$

- Example: line element in Euclidean space in Cartesian and spherical coordinates

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$x = r\sin\theta\cos\phi \rightarrow dx = \sin\theta\cos\phi\, dr + r\cos\theta\sin\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$$

$$y = r\sin\theta\sin\phi \rightarrow dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$$

$$z = r\cos\theta \rightarrow dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

$$\Rightarrow ds^2 = (\sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi)^2$$
$$+ (\sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi)^2$$
$$+ (\cos\theta\, dr - r\sin\theta\, d\theta)^2$$
$$= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

(Equivalent to $g_{a'b'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} g_{ab}$)

- It can be shown that at a given point p on M one can always find coordinates s.t. the metric g_{ab} takes the form

$$g_{ab} = \mathrm{diag}(-1, -1, \ldots, -1, 1, \ldots, 1, 0, \ldots, 0)$$

with $\partial_c g_{ab}\big|_p = 0$ but $\partial_c \partial_d g_{ab}\big|_p \neq 0$

- Signature: number of positive, negative and zero eigenvalues of g_{ab}
- Riemannian metric: all eigenvalues are positive

- Lorentzian: all positive and one negative (or all negative and one positive)
- Degenerate (or null): some eigenvalues are 0

Given a metric g_{ab} on M we can define:

- Norm of a vector V^a: $|V|^2 = g_{ab} V^a V^b$
- Inner product between two vectors: $A \cdot B = g_{ab} A^a B^b$
- Null vectors: $g_{ab} A^a A^b = 0$ (only for Lorentzian)
- Lower and raise indices:

$$V_a = g_{ab} V^b \quad \text{and} \quad W^a = g^{ab} W_b$$

and similarly for higher rank tensors.

- <u>Covariant derivatives</u>

- Want to define a new derivative operator, called covariant derivative and denoted by ∇, s.t.

  1) Is independent of the coordinates
  2) Maps (k,l) tensors to $(k, l+1)$ tensors

- Consider two arbitrary tensor fields S and T.

To uniquely determine ∇ we demand that ∇ obeys:

1) Linearity: $\nabla(T+S) = \nabla T + \nabla S$
2) Leibniz: $\nabla(T\otimes S) = (\nabla T)\otimes S + T\otimes(\nabla S)$

→ ∇ can be written as ∂ plus some linear transf.

In the case of a vector field V^a:

$$\nabla_a V^b = \partial_a V^b + \underbrace{\Gamma^b{}_{ac}}_{\text{connection coefficients.}} V^c$$

→ We can find the Γ's by demanding that $\nabla_a V^b$ transforms as a $(1,1)$ tensor under changes of coordinates:

$$\nabla_{a'} V^{b'} = \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b}\nabla_a V^b$$

$$= \partial_{a'} V^{b'} + \Gamma^{b'}{}_{a'c'} V^{c'}$$

$$= \frac{\partial x^a}{\partial x^{a'}}\partial_a\left(\frac{\partial x^{b'}}{\partial x^b} V^b\right) + \Gamma^{b'}{}_{a'c'}\frac{\partial x^{c'}}{\partial x^c} V^c$$

$$= \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b}\partial_a V^b + \frac{\partial x^a}{\partial x^{a'}}\frac{\partial^2 x^{b'}}{\partial x^a\partial x^b} V^b + \Gamma^{b'}{}_{a'c'}\frac{\partial x^{c'}}{\partial x^c} V^c$$

$$= \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b}\left(\partial_a V^b + \Gamma^b{}_{ac} V^c\right)$$

$$\Rightarrow \Gamma^{b'}{}_{a'c'} \frac{\partial x^{c'}}{\partial x^c} V^c + \frac{\partial x^a}{\partial x^{a'}} \frac{\partial^2 x^{b'}}{\partial x^a \partial x^b} V^b = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \Gamma^b{}_{ac} V^c$$

$$\Gamma^{b'}{}_{a'c'} \frac{\partial x^{c'}}{\partial x^c} + \frac{\partial x^a}{\partial x^{a'}} \frac{\partial^2 x^{b'}}{\partial x^a \partial x^c} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} \Gamma^b{}_{ac}$$

Multiply this equation by $\frac{\partial x^c}{\partial x^{d'}}$ and relabel $d' \to c'$

$$\Gamma^{b'}{}_{a'c'} = \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^c}{\partial x^{c'}} \frac{\partial x^{b'}}{\partial x^b} \Gamma^b{}_{ac} - \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^c}{\partial x^{c'}} \frac{\partial^2 x^b}{\partial x^a \partial x^c}$$

$\Rightarrow$ The Γ's are NOT tensors!!!

→ We further impose:

3) ∇ commutes with contractions: $\nabla_a (T^c{}_{cb}) = (\nabla T)_a{}^c{}_{cb}$

$$\Rightarrow \nabla_a \delta^b{}_c = 0$$

4) Reduces to the partial derivative when acting on scalars: $\nabla_a \phi = \partial_a \phi$

From these properties we can deduce the action of ∇ on 1-forms:

$$\begin{aligned}
\nabla_a (\omega_b V^b) &= (\nabla_a \omega_b) V^b + \omega_b \nabla_a V^b \\
&= (\nabla_a \omega_b) V^b + \omega_b (\partial_a V^b + \Gamma^b{}_{ac} V^c) \\
&= \partial_a (\omega_b V^b) = (\partial_a \omega_b) V^b + \omega_b \partial_a V^b
\end{aligned}$$

$$\Rightarrow \quad \nabla_a \omega_b = \partial_a \omega_b - \Gamma^c{}_{ab}\,\omega_c$$

Proceeding in the same way one can determine the formula for ∇ acting on an arbitrary (k,l) tensor:

$$\nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = \partial_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$$
$$+ \Gamma^{a_1}{}_{cd} T^{d a_2 \dots a_k}{}_{b_1 \dots b_l} + \Gamma^{a_k}{}_{cd} T^{a_1 \dots a_{k-1} d}{}_{b_1 \dots b_l}$$
$$- \Gamma^{d}{}_{c b_1} T^{a_1 \dots a_k}{}_{d b_2 \dots b_l} - \Gamma^{d}{}_{c b_l} T^{a_1 \dots a_k}{}_{b_1 \dots b_{l-1} d}$$

Alternative notation:

$$\nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = T^{a_1 \dots a_k}{}_{b_1 \dots b_l ; k}$$

- One can still define many connections satisfyin 1) − 4)
- The difference between two connections Γ and $\tilde{\Gamma}$ is a tensor:

$$\nabla_a V^b - \tilde{\nabla}_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c - (\partial_a V^b - \tilde{\Gamma}^b{}_{ac} V^c)$$
$$= (\Gamma^b{}_{ac} - \tilde{\Gamma}^b{}_{ac}) V^c$$

→ Since the LHS is a tensor by definition of covariant derivative, the RHS must also be a tensor:

$$\Gamma^b{}_{ac} - \tilde{\Gamma}^b{}_{ac} = S^b{}_{ac}$$

- Given a connection $\Gamma^a{}_{bc}$ one can define a new connection by permuting the lower indices
$\Gamma^a{}_{bc} \to \Gamma^a{}_{cb}$ also transforms as a connection

$\Rightarrow$ To every connection we can associate a torsion tensor:

$$T^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} = 2\,\Gamma^a{}_{[bc]}$$

- On a manifold with a metric g_{ab} we can define a unique connection by introducing the following two additional requirements on $\Gamma^a{}_{bc}$:

5) Torsion-free: $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$

6) Metric compatibility: $\nabla_a g_{bc} = 0$

$\Rightarrow$ Levi-Civita connection

$\rightarrow$ Torsion-free: $\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi$

$\rightarrow$ Metric compatibility:

$$0 = \nabla_a \delta^b{}_c = \nabla_a (g^{bd} g_{dc}) = g_{dc} \nabla_a g^{bd} + g^{bd} \overset{0}{\cancel{\nabla_a g_{dc}}}$$

$$= g_{dc} \nabla_a g^{bd} \quad \Rightarrow \quad \nabla_a g^{bd} = 0$$

→ A metric-compatible covariant derivative commutes with raising and lowering the indices:

$$g_{bc} \nabla_a V^c = \nabla_a (g_{bc} V^c) = \nabla_a V_b$$

· To show existence and uniqueness of the metric-compatible connection we will now find the Γ's:

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d{}_{ca} g_{db} - \Gamma^d{}_{cb} g_{ad} = 0$$
$$\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d{}_{ab} g_{dc} - \Gamma^d{}_{ac} g_{bd} = 0$$
$$\nabla_b g_{ca} = \partial_b g_{ca} - \Gamma^d{}_{bc} g_{da} - \Gamma^d{}_{ba} g_{cd} = 0$$

$$(1) - (2) - (3) = \partial_c g_{ab} - \partial_a g_{bc} - \partial_b g_{ac} + 2 \Gamma^d{}_{ab} g_{dc} = 0$$

Multiplying by g^{ec} and relabelling the indices,

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$$

→ Christoffel symbols

Example: Christoffel symbols of 2-dimensional flat space in polar coordinates.

$$ds^2 = g_{ab}\, dx^a dx^b = dr^2 + r^2 d\theta^2$$

$$\Rightarrow \quad g_{rr} = 1, \quad g_{\theta\theta} = r^2 \quad \Rightarrow \quad g^{rr} = 1, \quad g^{\theta\theta} = \frac{1}{r^2}$$

$$\Gamma^r{}_{rr} = \frac{1}{2} g^{rd} \left(\partial_r g_{rd} + \partial_r g_{rd} - \partial_d g_{rr} \right)$$
$$= \frac{1}{2} g^{rr} \left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr} \right) = 0$$

$$\Gamma^r{}_{\theta\theta} = \frac{1}{2} g^{rd} \left(2\, \partial_\theta g_{\theta d} - \partial_d g_{\theta\theta} \right)$$
$$= \frac{1}{2} g^{rr} \left(2\, \partial_\theta \cancelto{0}{g_{\theta r}} - \partial_r g_{\theta\theta} \right) = - r$$

Similarly we compute

$$\Gamma^r{}_{\theta r} = \Gamma^r{}_{r\theta} = 0$$

$$\Gamma^\theta{}_{rr} = 0, \quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r}, \quad \Gamma^\theta{}_{\theta\theta} = 0$$