MTH5113 (2023/24): Problem Sheet 3
Solutions

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(1) *(Warm-up)*

(a) Compute the integral

$$\int_0^1 f(x)\, dx,$$

where f is the real-valued function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = 1 + x + x^2 + x^3.$$

(b) Compute the integral

$$\int_{-\pi}^{\pi} g(t)\, dt,$$

where g is the real-valued function

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(t) = \sin t \cos t.$$

(c) Compute the double integral

$$\iint\limits_{\mathcal{R}} h\, dA,$$

where $\mathcal{R}$ is the rectangle $[0, 5] \times [0, 1]$, and where h is the function

$$h : \mathbb{R}^2 \to \mathbb{R}, \qquad h(x, y) = e^{2x} + e^x e^y.$$

(a) Notice that f is the derivative of the function

$$F : \mathbb{R} \to \mathbb{R}, \qquad F(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4,$$

that is, $F'(x) = f(x)$ for every $x \in \mathbb{R}$. Thus, by the fundamental theorem of calculus,

$$\begin{aligned} \int_0^1 f(x)\, dx &= \int_0^1 F'(x)\, dx \\ &= F(1) - F(0) \end{aligned}$$

$$= \left(1 + \frac{1}{2} \cdot 1^2 + \frac{1}{3} \cdot 1^3 + \frac{1}{4} \cdot 1^4\right) - \left(0 + \frac{1}{2} \cdot 0^2 + \frac{1}{3} \cdot 0^3 + \frac{1}{4} \cdot 0^4\right)$$
$$= \frac{25}{12}.$$

(b) The easiest method is to recall the trigonometric identity

$$g(t) = \sin t \cos t = \frac{1}{2} \sin(2t), \qquad t \in \mathbb{R}.$$

Noting that g is the derivative of the function

$$G(t) = -\frac{1}{4} \cos(2t),$$

the fundamental theorem of calculus then yields

$$\int_{-\pi}^{\pi} g(t)\, dt = \left[-\frac{1}{4} \cos(2t)\right]\Big|_{x=-\pi}^{x=\pi} = -\frac{1}{4} \cos(2\pi) + \frac{1}{4} \cos(-2\pi) = 0.$$

Alternatively, if you do not remember the double angle formula, you can also integrate g by substitution. In particular, applying the change of variables

$$u = \sin t, \qquad du = \cos t\, dt,$$

we then obtain

$$\int_{-\pi}^{\pi} g(t)\, dt = \int_{-\pi}^{\pi} \sin t \cos t\, dt = \int_{0}^{0} u\, du = 0.$$

In particular, we noticed that $t = \pm\pi$ both corresponded to $u = 0$.

(c) First, we apply Fubini's theorem to write the double integral as

$$\iint_{\mathcal{R}} h\, dA = \int_0^5 \left[\int_0^1 (e^{2x} + e^x e^y) dy\right] dx.$$

To evaluate the inner integral, we treat x as a constant and integrate with respect to y:

$$\iint_{\mathcal{R}} h\, dA = \int_0^5 \left(e^{2x} y + e^x e^y\right)\Big|_{y=0}^{y=1} dx$$
$$= \int_0^5 [e^{2x} + (e - 1)e^x] dx.$$

The remaining integral can also be computed directly:

$$\begin{aligned}
\iint_{\mathcal{R}} h\, dA &= \left[\frac{1}{2}e^{2x} + (e-1)e^{x}\right]_{x=0}^{x=5} \\
&= \left[\frac{1}{2}e^{10} + (e-1)e^{5}\right] - \left[\frac{1}{2} + (e-1)\right] \\
&= \frac{1}{2}e^{10} + e^{6} - e^{5} - e + \frac{1}{2}.
\end{aligned}$$

One can also integrate in the reverse order:

$$\begin{aligned}
\iint_{\mathcal{R}} h\, dA &= \int_0^1 \left[\int_0^5 (e^{2x} + e^{x}e^{y})\, dx\right] dy \\
&= \int_0^1 \left(\frac{1}{2}e^{10} + e^{5}e^{y} - \frac{1}{2} - e^{y}\right) dy \\
&= \frac{1}{2}e^{10} + e^{6} - e^{5} - e + \frac{1}{2}.
\end{aligned}$$

Both methods yield the same answer.

(2) *(Warm-up)*

(a) Consider the function

$$V : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}, \qquad V(x,y) = \ln(x^2 + y^2).$$

(i) Compute the gradient $\nabla V(x,y)$ for each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(ii) Find $\nabla V(3,4)$ and $\nabla V(-5,12)$.

(b) Consider the function

$$w : \mathbb{R}^3 \to \mathbb{R}, \qquad w(x,y,z) = xy + xz + yz.$$

(i) Compute the gradient $\nabla w(x,y,z)$ for each $(x,y,z) \in \mathbb{R}^3$.

(ii) Find $\nabla w(-1,1,6)$.

(a) These are direct computations using the definition of the gradient:

(i) First, we compute the partial derivatives of V. Using the chain rule yields

$$\partial_1 V(x,y) = \frac{1}{x^2+y^2} \cdot \partial_x(x^2+y^2) = \frac{2x}{x^2+y^2},$$
$$\partial_2 V(x,y) = \frac{1}{x^2+y^2} \cdot \partial_y(x^2+y^2) = \frac{2y}{x^2+y^2}.$$

Thus, by definition, the gradient of V, at any nonzero $(x,y) \in \mathbb{R}^2$, is

$$\nabla V(x,y) = (\partial_1 V(x,y), \partial_2 V(x,y))_{(x,y)} = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right)_{(x,y)}.$$

(ii) Here, we plug in the appropriate values for x and y:

$$\nabla V(3,4) = \left(\frac{2 \cdot 3}{3^2+4^2}, \frac{2 \cdot 4}{3^2+4^2} \right)_{(3,4)} = \left(\frac{6}{25}, \frac{8}{25} \right)_{(3,4)},$$
$$\nabla V(-5,12) = \left(\frac{2 \cdot (-5)}{5^2+12^2}, \frac{2 \cdot 12}{5^2+12^2} \right)_{(-5,12)} = \left(-\frac{10}{169}, \frac{24}{169} \right)_{(-5,12)}.$$

(b) These are again direct computations:

(i) Taking partial derivatives, we obtain

$$\partial_1 w(x,y,z) = y+z, \qquad \partial_2 w(x,y,z) = x+z, \qquad \partial_3 w(x,y,z) = x+y.$$

Thus, the gradient of w, at any $(x,y,z) \in \mathbb{R}^3$, is given by

$$\nabla w(x,y,z) = (y+z, x+z, x+y)_{(x,y,z)}.$$

(ii) Here, we plug in the appropriate values for x, y, and z:

$$\nabla w(-1,1,6) = (1+6, -1+6, -1+1)_{(-1,1,6)} = (7,5,0)_{(-1,1,6)}.$$

(3) *(Warm-up)* Are the following parametric curves regular?

(a) *Quartic function:*

$$\mathbf{a} : \mathbb{R} \to \mathbb{R}^3, \qquad \mathbf{a}(t) = (t, 0, t^4).$$

(b) *No idea what to call this thing:*

$$\mathbf{b} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{b}(t) = ((t-1)^3, e^{(t-1)^2}).$$

(c) *Lemniscate of Gerono:*

$$\mathbf{c} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{c}(t) = (\cos t, \sin t \cos t).$$

(a) To check whether $\mathbf{a}$ is regular, we compute its derivative for each $t \in \mathbb{R}$:

$$\mathbf{a}'(t) = (1, 0, 4t^3).$$

In particular, $\mathbf{a}'(t)$ never vanishes (since its x-component is always 1), hence $|\mathbf{a}'(t)|$ is everywhere non-zero. Thus, by definition, $\mathbf{a}$ is regular.

We can also check $|\mathbf{a}'(t)| \neq 0$ directly—in particular,

$$|\mathbf{a}'(t)| = \sqrt{1 + 16t^6} \geq \sqrt{1} = 1 \neq 0, \qquad t \in \mathbb{R},$$

since $16t^6$ is always non-negative.

(b) We begin by differentiating $\mathbf{b}$ (via the power and chain rules):

$$\mathbf{b}'(t) = (3(t-1)^2, 2(t-1)e^{(t-1)^2}).$$

Note in particular that

$$\mathbf{b}'(1) = (3(1-1)^2, 2(1-1)e^{(1-1)^2}) = (0, 0), \qquad |\mathbf{b}'(1)| = 0.$$

As a result, $\mathbf{b}$ is not regular.

(c) Differentiating $\mathbf{c}$ using the product rule, we obtain

$$\mathbf{c}'(t) = (-\sin t, \cos^2 t - \sin^2 t) = (-\sin t, 1 - 2\sin^2 t),$$

where in the last step, we used that $\cos^2 t + \sin^2 t = 1$.

Observe that given any $t \in \mathbb{R}$, whenever the x-component of $\mathbf{c}'(t)$ vanishes (that is, $\sin t = 0$), the y-component of $\mathbf{c}'(t)$ is $1 - 2\sin^2 t = 1 \neq 0$ and hence is nonzero. As a result,

$\mathbf{c}'(t)$ can never vanish for any $t \in \mathbb{R}$, and thus $\mathbf{c}$ is regular.

(4) [**Marked**] Let g be the function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x, y, z) = \frac{2}{625} e^{\frac{x+y}{5}} (x + y),$$

and let C denote the region

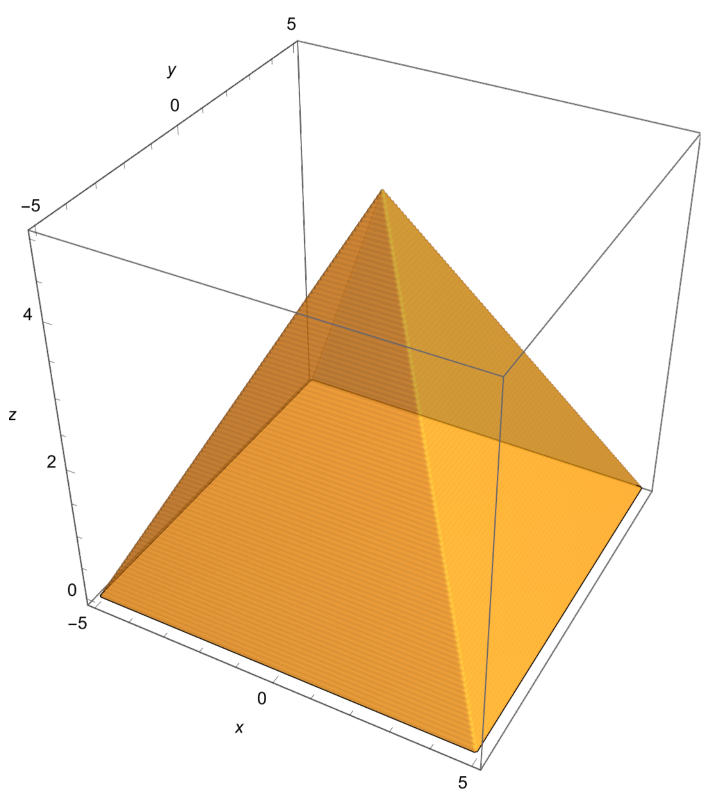
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid -(5 - z) \le x \le (5 - z),\ -(5 - z) \le y \le (5 - z),\ 0 \le z \le 5\}.$$

(a) Sketch the region C.

(b) Compute the triple integral

$$\iiint_C f \, dV.$$

(a) A sketch of the solid region C is given below: [2 mark for mostly correct sketch]



(b) We first apply Fubini's theorem to decompose

$$\iiint\limits_C f\,dV = \frac{2}{625}\int_0^5 \int_{-(5-z)}^{(5-z)} \int_{-(5-z)}^{(5-z)} e^{\frac{x+y}{5}}(x+y)\,dx\,dy\,dz. \text{ [1 mark]}$$

The inner integral can now be computed using the fundamental theorem of calculus:

$$\begin{aligned}\iiint\limits_C f\,dV &= \frac{2}{125}\int_0^5 \int_{-(5-z)}^{(5-z)} e^{\frac{x+y}{5}}(x+y-5)\Big|_{x=-(5-z)}^{x=5-z}\,dy\,dz \\ &= -\frac{2}{125}\int_0^5 \int_{-(5-z)}^{(5-z)} e^{\frac{1}{5}(y-z-5)}\left(e^2(z-y)+e^{2z/5}(y+z-10)\right)dy\,dz\end{aligned}$$

Applying the fundamental theorem of calculus again yields

$$\begin{aligned}\iiint\limits_C f\,dV &= \int_0^5 -\frac{2}{25}e^{\frac{1}{5}(y-z-5)}\left(e^2(-y+z+5)+e^{2z/5}(y+z-15)\right)\Big|_{y=-(5-z)}^{y=5-z}\,dz \\ &= \frac{4}{25}\int_0^5 \left(e^{\frac{2z}{5}-2}(z-10) - z\,e^{2-\frac{2z}{5}} + 10\right)dz.\end{aligned}$$

The final integral is similarly computed:

$$\begin{aligned}\iiint\limits_C f\,dV &= \frac{1}{5}\left(8z + e^{\frac{2z}{5}-2}(2z-25) + e^{2-\frac{2z}{5}}(2z+5)\right)\Big|_{z=0}^{z=5} \\ &= 8 + \frac{5}{e^2} - e^2.\end{aligned}$$

[2 marks for mostly correct computation]

(5) **[Tutorial]** Answer the following:

(a) Let f be the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) = x^2 y,$$

and let D denote the triangular region

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ |x| \le y\}.$$

(i) Sketch the region D on a Cartesian plane.

(ii) Compute the double integral

$$\iint\limits_D f\,dA.$$

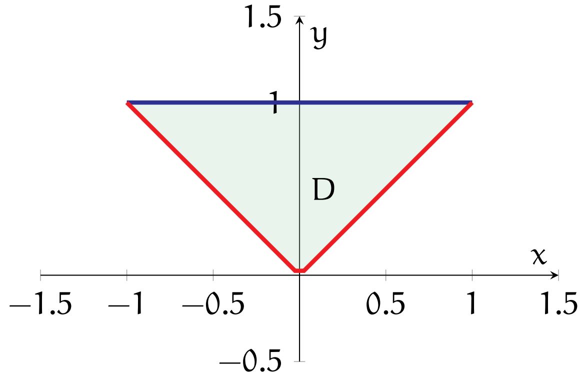
(b) Let Q denote the region

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq y + z,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}.$$

(i) Sketch the region Q (or at least, do the best you can).

(ii) Use a triple integral to compute the volume of Q.

(a) A sketch of D is below (D is the green region):



To compute the double integral, we apply Fubini's theorem (to a rectangular region containing D, and to a function that is equal to f on D and vanishes outside D):

$$\iint_D f\, dA = \int_0^1 \left[\int_{-y}^{y} x^2 y\, dx \right] dy.$$

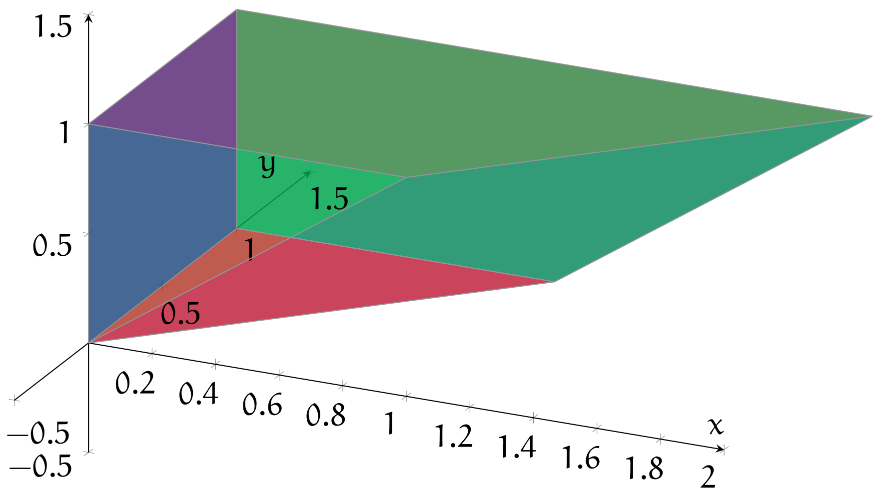
To evaluate the inner integral, we apply the fundamental theorem of calculus:

$$\iint_D f\, dA = \frac{1}{3} \int_0^1 y(x^3)\big|_{x=-y}^{x=y} dy = \frac{2}{3} \int_0^1 y^4 dy.$$

Applying the fundamental theorem of calculus again to the remaining integral yields

$$\iint_D f\, dA = \frac{2}{3} \cdot \frac{1}{5} y^5 \bigg|_{y=0}^{y=1} = \frac{2}{15}.$$

(b) A sketch of the solid region Q is given below:



To compute the volume of Q, we first apply Fubini's theorem to decompose

$$\mathcal{V}(Q) = \iiint_Q 1 \, dV = \int_0^1 \int_0^1 \int_0^{y+z} dx\,dy\,dz.$$

The iterated integrals can now be computed using the fundamental theorem of calculus:

$$\begin{aligned}\mathcal{V}(Q) &= \int_0^1 \int_0^1 (y+z)\,dy\,dz \\ &= \int_0^1 \left(\frac{1}{2} + z\right) dz \\ &= 1.\end{aligned}$$

(6) *(Fun with cycloids)* Consider the parametric curve

$$\mathbf{c} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{c}(t) = (t - \sin t, 1 - \cos t).$$

(The path mapped out by $\mathbf{c}$ is known as a *cycloid*.)

(a) Show that $\mathbf{c}$ is not regular. At which $t \in \mathbb{R}$ do the values $|\mathbf{c}'(t)|$ vanish?

(b) Plot the image of $\mathbf{c}$ using a computer *(see the links on the QMPlus page)*. What happens at the points $\mathbf{c}(t)$ along the plot at which $|\mathbf{c}'(t)| = 0$?

(a) Taking a derivative of $\mathbf{c}$ yields

$$\mathbf{c}'(t) = (1 - \cos t, \sin t).$$

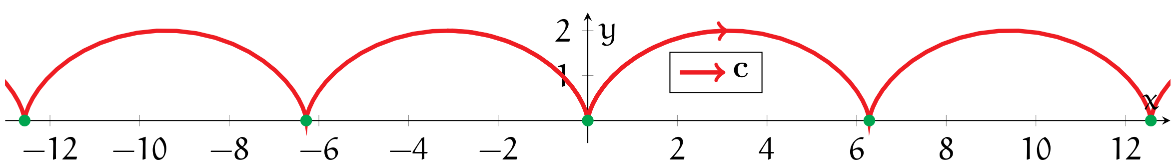
Taking the norm of the above, we see that

$$\begin{aligned} |\mathbf{c}'(t)| &= \sqrt{(1-\cos t)^2 + \sin^2 t} \\ &= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= \sqrt{2 - 2\cos t}. \end{aligned}$$

In particular, note that $|\mathbf{c}'(t)|$ vanishes whenever $\cos t = 1$.

Recalling the basic properties of the cosine function, we conclude that $|\mathbf{c}'(t)|$ vanishes whenever $t = 2k\pi$ for any integer k. In particular, $\mathbf{c}$ fails to be regular.

(b) A computer plot of the values of $\mathbf{c}$ is given below:



The points on the plot at which $\mathbf{c}'$ vanishes are marked in green. At these points, the plot contains a "jagged edge" in which the direction of the path changes instantaneously.

(7) *(More parametric curves)* For each of the following parametric curves γ: (i) sketch, with the help of a computer, the image of γ, and (ii) determine whether γ is regular.

(a) *Cissoid of Diocles:*

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \left(\frac{t^2}{1+t^2}, \frac{t^3}{1+t^2} \right).$$

(b) *Witch of Agnesi:*

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \left(t, \frac{1}{1+t^2} \right).$$

(c) *Tricuspoid:*

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (2\cos t + \cos(2t),\, 2\sin t - \sin(2t)).$$

(a) First, we differentiate γ using the quotient rule:

$$\begin{aligned}\gamma'(t) &= \left(\frac{d}{dt}\left(\frac{t^2}{1+t^2}\right), \frac{d}{dt}\left(\frac{t^3}{1+t^2}\right)\right)\\ &= \left(\frac{(1+t^2)\cdot 2t - t^2 \cdot 2t}{(1+t^2)^2}, \frac{(1+t^2)\cdot 3t^2 - t^3 \cdot 2t}{(1+t^2)^2}\right)\\ &= \left(\frac{2t}{(1+t^2)^2}, \frac{t^4+3t^2}{(1+t^2)^2}\right).\end{aligned}$$

Note in particular that

$$\gamma'(0) = \left(\frac{2\cdot 0}{(1+0^2)^2}, \frac{0^4 + 3\cdot 0^2}{(1+0)^2}\right) = (0,0).$$

As a result, γ is not regular.

(b) Differentiating γ yields, for each $t \in \mathbb{R}$,

$$\gamma'(t) = \left(1, -\frac{2t}{(1+t^2)^2}\right).$$

In particular, observe that $\gamma'(t) \neq (0,0)$ for any $t \in \mathbb{R}$, since the x-component of $\gamma'(t)$ is never vanishes. Thus, γ is regular.

(c) First, differentiating γ yields

$$\gamma'(t) = (-2\sin t - 2\sin(2t),\, 2\cos t - 2\cos(2t)).$$

Observe in particular that,

$$\gamma'(0) = (-2\cdot 0 - 2\cdot 0,\, 2\cdot 1 - 2\cdot 1) = (0,0),$$

and hence γ fails to be regular.

If you cannot see the above directly, you can also try to directly solve the system

$$-2\sin t - 2\sin(2t) = 0, \qquad 2\cos t - 2\cos(2t) = 0. \tag{1}$$

Using some trigonometric identities, the first equation of (1) can be rearranged as

$$-\sin t = \sin(2t) = 2\sin t \cos t,$$

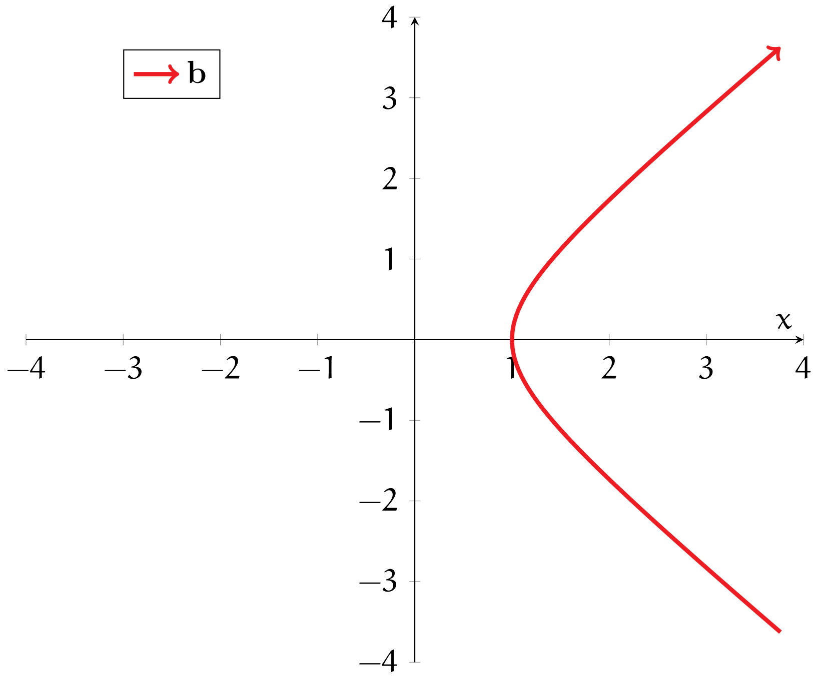
which is satisfied if and only if $\cos t = -\frac{1}{2}$ or $\sin t = 0$. One specific solution of this is $t = 0$, which you can then check also satisfies the second equation in (1). (Other values of t that also solve both equations in (1) include $t = \frac{2\pi}{3}$ and $t = \frac{4\pi}{3}$.)

(8) *(Reparametrise my hyperbola!)* Consider the following parametric curves:

$$\mathbf{a} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{a}(t) = (\cosh t, \sinh t),$$
$$\mathbf{b} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{b}(t) = \left(\sqrt{1 + t^2}, t\right).$$

- **(a)** Sketch the image of **b**.
- **(b)** Show that both **a** and **b** are regular.
- **(c)** Show that $\mathbf{a}(t) = \mathbf{b}(\sinh t)$ for any $t \in \mathbb{R}$. According to definition, what else must you to show in order to demonstrate that **a** and **b** are reparametrisations of each other?
- **(d)** Finish what you started in (c)—show that **a** and **b** are reparametrisations of each other. *(You will not need advanced knowledge, but you will have to be extra resourceful.)*

(a) A sketch of the image of **b** is found below:



(b) First, for $\mathbf{a}$, we recall the derivative formulas for cosh and sinh:

$$\mathbf{a}'(t) = (\sinh t, \cosh t), \qquad t \in \mathbb{R}.$$

Recalling the identity $\cosh^2 t - \sinh^2 t = 1$, we obtain, for each $t \in \mathbb{R}$,

$$|\mathbf{a}'(t)| = \sqrt{\sinh^2 t + \cosh^2 t} = \sqrt{1 + 2\sinh^2 t} \geq \sqrt{1} > 0,$$

and it follows that $\mathbf{a}$ is indeed regular.

Similarly, for $\mathbf{b}$, we differentiate:

$$\mathbf{b}'(t) = \left(\frac{t}{\sqrt{1+t^2}}, 1 \right).$$

Since the y-component of $\mathbf{b}'(t)$ never vanishes, it follows that $\mathbf{b}$ is regular.

(c) Using again that $\cosh^2 t - \sinh^2 t = 1$, we compute

$$\mathbf{b}(\sinh t) = \left(\sqrt{1 + \sinh^2 t}, \sinh t \right) = \left(\sqrt{\cosh^2 t}, \sinh t \right) = (\cosh t, \sinh t) = \mathbf{a}(t),$$

where in the second to last step, we recalled that $\cosh t$ is always positive.

To show that $\mathbf{a}$ and $\mathbf{b}$ are reparametrisations of each other, we must show, in addition to the above, that the change of variables $\phi(t) = \sinh t$ satisfies: (i) ϕ is smooth, (ii) ϕ is a bijection between $\mathbb{R}$ and itself, and (iii) its inverse ϕ^{-1} is smooth.

(d) First, note that ϕ is smooth since

$$\phi(t) = \sinh t = \frac{1}{2}(e^t - e^{-t}),$$

and the exponential functions on the right-hand side are clearly smooth.

Next, we recall that ϕ is always strictly increasing, since for any $t \in \mathbb{R}$,

$$\phi'(t) = \cosh t = \frac{1}{2}(e^t + e^{-t}) > 0.$$

In particular, if $t < t'$, then $\phi(t) < \phi(t')$. Thus, it follows that ϕ is injective.

Now, consider any $s \in \mathbb{R}$, and let us try to solve

$$\sinh t = \phi(t) = s.$$

~~Consulting Google~~ Applying some really clever algebraic manipulations, the above becomes

$$s = \frac{1}{2}(e^t - e^{-t}), \qquad (e^t)^2 - 2s \cdot e^t - 1 = 0,$$

and the quadratic formula yields

$$e^t = s \pm \sqrt{s^2 + 1}.$$

Since $s + \sqrt{s^2 + 1} > 0$ (for any $s \in \mathbb{R}$), we can take its logarithm, and hence

$$t = \ln\left(s + \sqrt{s^2 + 1}\right)$$

solves the equation $\phi(t) = s$. In particular, ϕ is surjective onto $\mathbb{R}$. Moreover, since ϕ is injective and surjective, it follows that ϕ is a bijection between $\mathbb{R}$ and itself.

Finally, the above derivation also gives a formula for the inverse of ϕ:

$$\phi^{-1}(s) = \ln\left(s + \sqrt{s^2 + 1}\right).$$

Since $s + \sqrt{s^2 + 1} > 0$ for all $s \in \mathbb{R}$, and since ln is infinitely differentiable as long as its input is positive, it follows that ϕ^{-1} is smooth.

(9) *(Numbers, Sets, and Functions revisited)* Let $\mathcal{P}$ denote the set of all regular parametric curves in $\mathbb{R}^n$. Given any two $\gamma_1, \gamma_2 \in \mathcal{P}$, we write $\gamma_1 \sim \gamma_2$ iff γ_1 is a reparametrisation of γ_2. Show that this $\sim$ defines an *equivalence relation* on $\mathcal{P}$.

To show $\sim$ is an equivalence relation, we must show $\sim$ is *reflexive*, *symmetric*, and *transitive*.

First, to show $\sim$ is *reflexive*, we must show that $\gamma \sim \gamma$ *for any* $\gamma \in \mathcal{P}$. To see this, we simply note that if $\gamma : I \to \mathbb{R}^n$, then the identity function on I,

$$\phi_0 : I \to I, \qquad \phi(t) = t,$$

trivially satisfies $\gamma(\phi_0(t)) = \gamma(t)$ for all $t \in I$. Moreover, clearly ϕ_0 is a bijection between I and itself, and both ϕ_0 and its inverse (which is equal to ϕ_0) are smooth as well. Thus, by definition, γ is a reparametrisation of itself, and hence $\gamma \sim \gamma$.

Next, to show *symmetry*, we must show that *if* $\gamma_1 \sim \gamma_2$*, where* $\gamma_1 : I_1 \to \mathbb{R}^n$ *and* $\gamma_2 : I_2 \to \mathbb{R}^n$ *are regular parametric curves, then* $\gamma_2 \sim \gamma_1$ *as well.* Since we assume $\gamma_1 \sim \gamma_2$, the corresponding (bijective) change of variables $\phi : I_1 \leftrightarrow I_2$ satisfies $\gamma_2(\phi(t)) = \gamma_1(t)$ for all $t \in I_1$, with both ϕ and ϕ^{-1} being smooth. A direct substitution then yields

$$\gamma_1(\phi^{-1}(t)) = \gamma_2(t), \qquad t \in I_2.$$

Moreover, ϕ^{-1} is clearly a bijection between I_2 and I_1, and both ϕ^{-1} and its inverse ϕ are already known to be smooth. Thus, γ_2 is a reparametrisation of γ_1, that is, $\gamma_2 \sim \gamma_1$.

Finally, to show $\sim$ is *transitive*, we must show that *if* $\gamma_1 \sim \gamma_2$ *and* $\gamma_2 \sim \gamma_3$*, where* γ_1 *and* γ_2 *are as before, and where* $\gamma_3 : I_3 \to \mathbb{R}^n$ *is a regular parametric curve, then* $\gamma_1 \sim \gamma_3$. Now, let $\phi_{12} : I_1 \leftrightarrow I_2$ and $\phi_{23} : I_2 \leftrightarrow I_3$ denote the corresponding changes of variables, satisfying

$$\gamma_2(\phi_{12}(t)) = \gamma_1(t), \quad t \in I_1, \qquad \gamma_3(\phi_{23}(s)) = \gamma_2(s), \quad s \in I_2,$$

and let ϕ_{13} be the *composition* $\phi_{23} \circ \phi_{12}$ of ϕ_{23} with ϕ_{12}. Then, ϕ_{13} is a bijection between I_1 and I_3, and by the chain rule, both $\phi_{13} = \phi_{23} \circ \phi_{12}$ and its inverse $\phi_{13}^{-1} = \phi_{12}^{-1} \circ \phi_{23}^{-1}$ are smooth. Furthermore, a direct computation yields that

$$\gamma_3(\phi_{13}(t)) = \gamma_3(\phi_{23}(\phi_{12}(t))) = \gamma_2(\phi_{12}(t)) = \gamma_1(t), \qquad t \in I_1.$$

Combining all the above, we conclude that γ_1 is a reparametrisation of γ_3, and thus $\gamma_1 \sim \gamma_3$.