MTH5113 (2023/24): Problem Sheet 5

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 1**.

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(1) *(Warm-up)* Given a curve C, along with a pair of parametrisations γ_1 and γ_2 of C:

(i) Compute, for each parameter t, the unit tangent vectors

$$\frac{1}{|\gamma_1'(t)|} \cdot \gamma_1'(t)_{\gamma_1(t)}, \qquad \frac{1}{|\gamma_2'(t)|} \cdot \gamma_2'(t)_{\gamma_2(t)}.$$

(ii) Determine whether γ_1 and γ_2 generate the same orientation or opposite orientations of C. Give a brief justification of your answer.

(a) *Circle:* $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and

$$\gamma_1 : \mathbb{R} \to C, \qquad \gamma_1(t) = (\cos t, \sin t),$$
$$\gamma_2 : \mathbb{R} \to C, \qquad \gamma_2(t) = (-\sin t, -\cos t).$$

(b) *Line:* $C = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 2,\ x + z = 1\}$, and

$$\gamma_1 : \mathbb{R} \to C, \qquad \gamma_1(t) = (t,\ 2 - t,\ 1 - t),$$
$$\gamma_2 : \mathbb{R} \to C, \qquad \gamma_2(t) = (2 - t,\ t,\ t - 1).$$

(2) *(Warm-up)* Find both the unit tangents and the unit normals to each of the following curves C at the given point $\mathbf{p}$.

(a) $C = \{(t^3, t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}$ and $\mathbf{p} = (-8, -2)$.

(b) $C = \{(x, y) \in \mathbb{R}^2 \mid x^4 + 2y^2 = 3\}$ and $\mathbf{p} = (-1, 1)$.

(3) *(Warm-up)* Consider the following regular parametric curve:

$$\mathbf{b} : (0,1) \to \mathbb{R}^2, \qquad \mathbf{b}(t) = \left(t, \frac{2}{3} t^{\frac{3}{2}}\right).$$

(a) Compute the arc length of $\mathbf{b}$.

(b) Compute the curve integral of the function F over $\mathbf{b}$, where

$$F : \mathbb{R}^2 \to \mathbb{R}, \qquad F(x, y) = 1 + x.$$

(c) Compute the curve integral of the function G over $\mathbf{b}$, where

$$G : \mathbb{R} \times (0, \infty) \to \mathbb{R}, \qquad G(x, y) = \frac{2}{3} + \frac{y}{\sqrt{x}}.$$

(4) [**Marked**] Consider the following curve which traces out a quarter peanut:

$$C = \left\{ (x, y) \in \mathbb{R}^2 \mid \left(\left(x - \frac{3}{2}\right)^2 + y^2\right)\left(\left(x + \frac{3}{2}\right)^2 + y^2\right) = 9,\ x > 0,\ y > 0 \right\},$$

and consider the following function,

$$F : \mathbb{R}^2 \to \mathbb{R}, \qquad F(x, y) = \frac{2xy}{3\sqrt{x^2 + y^2}}.$$

(a) Consider the following function γ:

$$\gamma(t) = \left(\sqrt{(a + \cos t)(b + \cos t)},\ \sin t\right) .$$

For which a and b and for what domain is γ a *bijective* parametrisation of C such that *the image of* γ *differs from* C *by only a finite number of points*?

(b) Compute the curve integral of F over C.

(5) [**Tutorial**] Let us derive some (possibly) familiar formulas!

(a) Let $\mathcal{C}$ denote the unit circle,

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Show that at each $\mathbf{p} \in \mathcal{C}$, the unit normals to $\mathcal{C}$ at $\mathbf{p}$ are precisely $\pm \mathbf{p}_{\mathbf{p}}$.

(b) Let G_f denote the graph of a function $f : (a, b) \to \mathbb{R}$:

$$\mathsf{G}_f = \{(x, y) \in \mathbb{R}^2 \mid y = f(x),\ a < x < b\}.$$

Find a formula for the arc length of G_f.

(c) Let $\rho : (c, d) \to \mathbb{R}$, and consider the following *polar parametric curve*:

$$\lambda_\rho : (c, d) \to \mathbb{R}^2, \qquad \lambda_\rho(\theta) = (\rho(\theta)\, \cos\theta,\ \rho(\theta)\, \sin\theta).$$

Find a formula for the arc length of λ_ρ.

(6) *(Issues with parametrisations)* Let H denote the curve

$$\mathsf{H} = \{(x, y, z) \in \mathbb{R}^3 \mid x = \cosh z,\ y = \sinh z\},$$

and let γ be the parametric curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (\cosh t,\ \sinh t,\ t).$$

(For this problem, you may assume that you already know H *is a curve.)*

(a) What statements must you prove in order to show that γ is a parametrisation of H, according to the definition given in this module?

(b) Show that γ is indeed a parametrisation of H.

(c) Oh no, Mr. Error (from question (6) of *Problem Sheet 4*) is back to his erroneous ways! He decides to describe the points of H using the parametric curve

$$\zeta : \mathbb{R} \to \mathsf{H}, \qquad \zeta(t) = (\cosh t^2,\ \sinh t^2,\ t^2).$$

He computes (correctly) that

$$\zeta(0) = (1, 0, 0) \in \mathsf{H}, \qquad \zeta'(0) = (0, 0, 0).$$

He then (incorrectly) concludes that $T_{(1,0,0)}\mathsf{H}$ contains only one element,

$$T_{(1,0,0)}\mathsf{H} = T_\zeta(0) = \{s \cdot \zeta'(0)_{\zeta(0)} \mid s \in \mathbb{R}\} = \{(0, 0, 0)_{(1,0,0)}\},$$

hence it is a 0-dimensional space! What error did Mr. Error make this time?

(7) *(Orient my hyperbola!)* Let $\mathcal{H}$ denote the *hyperbola*,

$$\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}.$$

Describe all the possible orientations of $\mathcal{H}$. How many such orientations are there?

(8) *(Arc length parametrisations)* Let (a, b) be a finite open interval, and let $\gamma : (a, b) \to \mathbb{R}^n$ be a regular parametric curve. We can then define the change of variables

$$s = \phi(t) = \int_a^t |\gamma'(\tau)|\, d\tau.$$

Note that s *represents the total length travelled by* γ *up to parameter* t.

(a) Show that the following holds for any $t \in (a, b)$:

$$\frac{ds}{dt} = |\gamma'(t)|.$$

The reparametrisation λ of γ defined as

$$\lambda : (0, L(\gamma)) \to \mathbb{R}^n, \qquad \lambda(s) = \gamma(t)$$

is called the *arc length reparametrisation*, since its parameter is itself the distance travelled.

(b) Let $R > 0$, and let γ be the regular parametric curve

$$\gamma : (0, 2\pi) \to \mathbb{R}^2, \qquad \gamma(t) = (R \cos t,\ R \sin t).$$

Find the arc length reparametrisation λ of this γ. What is the domain of λ?

(9) *(Parental advisory, implicit content)* *(Not examinable)* A special case of the *implicit function theorem* for functions of two variables can be stated as follows:

Theorem. *(Implicit Function Theorem)* Let $U \subseteq \mathbb{R}^2$ be open and connected, let $f : U \to \mathbb{R}$ be smooth, and let C denote the level set

$$C = \{(x, y) \in U \mid f(x, y) = c\}, \qquad c \in \mathbb{R}.$$

Suppose, in addition, that $(x, y) \in C$ and that $\partial_2 f(x, y) \neq 0$. Then, there exists some open set $V \subseteq \mathbb{R}^2$, with $(x, y) \in V$, such that $C \cap V$ is the graph of a function, i.e.,

$$C \cap V = \{(x, h(x)) \mid x \in I\},$$

where I is an open interval, and where $h : I \to \mathbb{R}$ is smooth.

In addition, an analogous theorem holds with the roles of x and y interchanged.

(a) How does the above implicit function theorem relate to the process of implicit differentiation that you learned in calculus? (An informal description will suffice here.)

(b) How does the above implicit function theorem relate to the proof of Theorem 3.26 in the lecture notes? Again, an informal description will suffice here.