**Question on Value at Risk and Expected Shortfall**

1. A gardener has a rare orchid plant which flowers only once a year. The flowers produced by the plant could be of any shade of pink but red flowers are considered bad outcomes by the gardener. The number of red flowers the orchid produces follows a Poisson distribution with a mean and variance of 8.

   Note that Poisson distribution is a discrete probability distribution that indicates the probability of a given number of events occurring in a fixed interval of time. A discrete random variable $X$ is said to have a Poisson distribution with parameter $\lambda > 0$ if, for $k = 0,1,2,\ldots$, the probability mass function of $X$ is given by $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ with $E(X) = Var(X) = \lambda$.

   i) Determine the Value at Risk for the number of flowers produced over the flowering year with 90% confidence interval.

   Answer:

   Value at risk for discrete random variable:
   $VaR(x; p) = -L$ where $L = \max\{x_i : Pr(X < x_i) \leq p$

   The red flowers are considered the losses on the distribution of returns.

   Looking at the Poisson distribution table we see that $Pr(X) \leq 4 = 0.09963$ and $Pr(X) \leq 5 = 0.191236$.
   Hence,

   $$19.1\% = 0.191236 = Pr(X \leq 5) < \boldsymbol{Pr(X < 5)} = Pr(X \leq 4) = 0.09963 \leq 0.1 = 10\%$$

   Hence VaR with 90% CI is 5 red flowers.
   The students could also calculate $Pr(X) \leq 5$ and $Pr(X) \leq 4$ using the Poisson probability mass formula:

   $$P(X = 0) = \frac{8^0 e^{-8}}{0!} = 0.00034$$

   $$P(X = 1) = \frac{8^1 e^{-8}}{1!} = 0.00268$$

   $$P(X = 2) = \frac{8^2 e^{-8}}{2!} = 0.01073$$

   $$P(X = 3) = \frac{8^3 e^{-8}}{3!} = 0.02863$$

   $$P(X = 4) = \frac{8^4 e^{-8}}{4!} = 0.05725$$

   $$P(X = 5) = \frac{8^5 e^{-8}}{5!} = 0.09160$$

   ii) Determine the gardener's expected shortfall over the flowering year with 90% confidence level.

   Answer:
   Expected shortfall is the expected loss when things have gone bad : VaR is 5 red flowers with 90% CL.

Expected loss: $P(X=0)(5-0)+P(X=1)(5-1)+P(X=2)(5-2)+P(X=3)(5-3)+P(X=4)(5-4)=0.002+0.011+0.032+0.057+0.057=0.159$

Hence the expected shortfall with 90%CI is 0.159 red flower.