First coursework due by 9am Mon 19 Feb
Submit on QMplus.

Recap quiz

Given an LP what is an extreme point solution

(i) intuitively/geometrically? corner of feasible region

(ii) formally? $\underline{x}$ is an extreme point solution if $\underline{x}$ is feasible and $\underline{x}$ cannot be written as convex combination $\lambda \underline{y} + (1-\lambda)\underline{z}$ of two other feasible solutions $\underline{y}$ and $\underline{z}$ ($\underline{y}, \underline{z} \neq \underline{x}$).

What are the 3 main steps in transforming an LP to standard inequality form?

Standard inequality form

$$\max \ \underline{c}^T \underline{x}$$

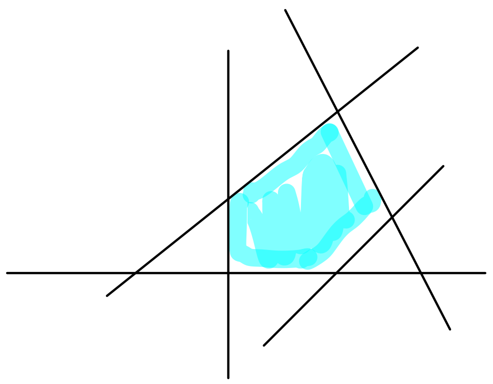
$$\text{sub to } \ A\underline{x} \leq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

fix goal

③ fix constraints

① fix sign restrictions



How to transform any linear program to standard inequality form

① For each variable x_i if sign constraint is

$x_i \geq 0$ ✓

$x_i \leq 0$ replace x_i with $\bar{x}_i \geq 0$ where $x_i = -\bar{x}_i$

x_i unrestricted replace x_i with $x_i^+ - x_i^-$

with $x_i^+ \geq 0$ $x_i^- \geq 0$

② If goal is min $\underline{c}^T\underline{x}$ replace with max $(-\underline{c})^T\underline{x}$

③ For each constraint, if constraint is

$\underline{a}^T\underline{x} \leq b$ ✓

$\underline{a}^T\underline{x} \geq b$ replace with $(-\underline{a})^T\underline{x} \leq -b$

$\underline{a}^T\underline{x} = b$ replace with two constraints

$\underline{a}^T\underline{x} \leq \underline{b}$

$(-\underline{a})^T\underline{x} \leq -b$

__Def__ We say an LP is in __standard equation form__ if it can be written as

$$\begin{aligned} \text{maximise} \quad & \underline{c}^T \underline{x} \\ \text{subject to} \quad & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

where A is an $m' \times n'$ matrix
$\underline{c} \in \mathbb{R}^{n'}$
$\underline{b} \in \mathbb{R}^{m'}$

Not the same A, $\underline{c}$, $\underline{b}$, $\underline{x}$ from Standard inequality form.

Every LP can be transformed into standard equation form.

We follow the same steps as we did for transforming into Standard inequality form except
- we leave equations as they are (in step 3)
- we add an extra step at the end

4 for each constraint of the form

$$\underline{a}^T \underline{x} \leq b \quad \text{i.e.} \quad a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b$$

We introduce a new slack variable s with $s \geq 0$ and replace constraint above with

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + s = b$$

Example Transform following LP into standard equation form.

$$\begin{aligned} \text{maximise} \quad & 2x_1 + 3x_2 \\ \text{sub to} \quad & 2x_1 - x_2 \leq 3 \\ & x_1 - 2x_2 \geq -6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

ans: ① and ② nothing to do.

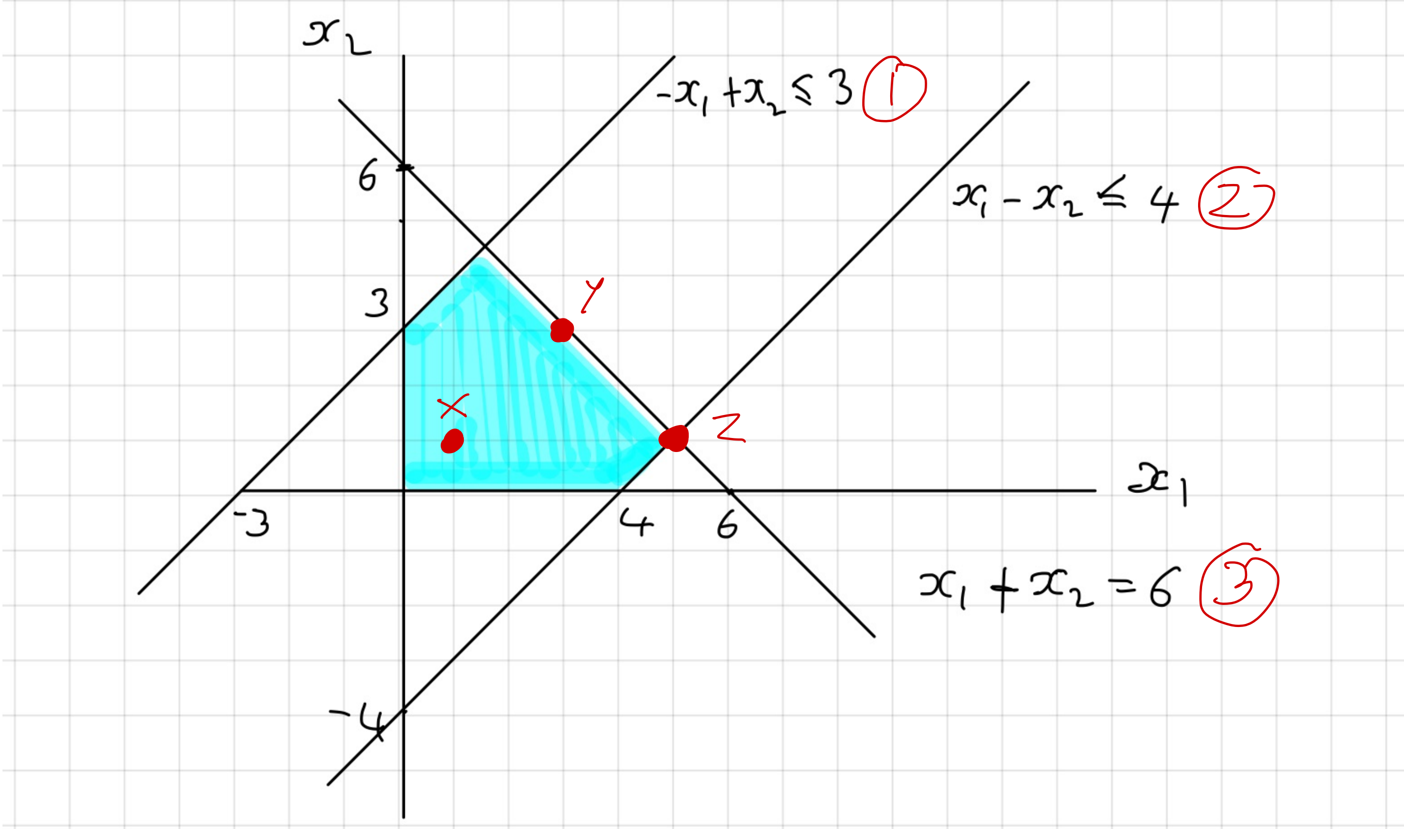
③
$$\begin{aligned} \text{maximise} \quad & 2x_1 + 3x_2 \\ \text{sub to} \quad & 2x_1 - x_2 \leq 3 \\ & -x_1 + 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

④
$$\begin{aligned} \text{maximise} \quad & 2x_1 + 3x_2 \\ \text{sub to} \quad & 2x_1 - x_2 + s_1 = 3 \\ & -x_1 + 2x_2 + s_2 = 6 \\ & x_1, x_2, s_1, s_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & \underline{c}^T \underline{x} \\ \text{sub to} \quad & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

$$\underline{c} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$



Task : (i) Give standard equation form
(ii) Write down feasable solution of standard equation form that correspond to X, Y, Z

standard inequality form

maximise $2x_1 + 3x_2$

sub to
$$-x_1 + x_2 \leq 3 \quad (1)$$
$$x_1 - x_2 \leq 4 \quad (2)$$
$$x_1 + x_2 \leq 6 \quad (3)$$
$$x_1, x_2 \geq 0$$

standard equation form

max $2x_1 + 3x_2$

sub to
$$-x_1 + x_2 + s_1 = 3$$
$$x_1 - x_2 + s_2 = 4$$
$$x_1 + x_2 + s_3 = 6$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

$$\begin{array}{cccc} & X & Y & Z \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 3 \\ 3 \end{pmatrix} & \begin{pmatrix} 5 \\ 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cccc} & X & Y & Z \\ \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = & \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} & \begin{pmatrix} 3 \\ 3 \\ 3 \\ 4 \\ 0 \end{pmatrix} & \begin{pmatrix} 5 \\ 1 \\ 7 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

Important remarks

Given LP

- each feasible solution of standard inequality form corresponds uniquely to a feasible solution of standard equation form.

- In standard inequality form a tight constraint is one that holds with equality.
  The corresponding slack variable is zero in standard equation form.

e.g. y constraint 3 is tight and $s_3 = 0$

z constraint 2, 3 tight and $s_2 = s_3 = 0$

- Informaly, expect extreme points to be the feasible solutions with many tight constraints, i.e. many variables equal to zero in standard equation form.

- If we can find optimal solution to transformed LP then can find optimal solution to original

(how?)

Recap quiz

Given an LP what is an extreme point solution

(i) intuitively/geometrically?

(ii) formally?

Given an LP in standard equation form

$$\begin{aligned} \max \quad & \underline{c}^T \underline{x} \\ \text{subject to} \quad & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

What is an optimal solution

(i) intuitively? best possible feasible solution

(ii) formally? $\underline{x}$ is optimal if it is feasible and $\underline{c}^T\underline{x} \geq \underline{c}^T\underline{x}'$ for any feasible $\underline{x}'$

Suppose $x \leq 100$ and $y \leq 100$

and average of x and y is exactly 100.

What can we say about x and y?

Then $x = y = 100$

<u>Thm</u> Every LP in standard equation form that has an optimal solution also has an optimal solution that is an extreme point solution.

<u>Pf</u> Assume LP is maximise $\underline{c}^T\underline{x}$
sub to $A\underline{x}=\underline{b}$, $\underline{x} \geq \underline{0}$.

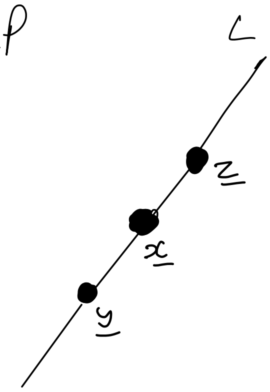
Suppose $\underline{x}$ is an optimal solution but not an extreme point solution

Aim: find optimal solution that is "more" extreme i.e. more zero entries.

Prove theorem in steps.

<u>Claim 1</u> $\underline{x}$ can be written as a convex combination of $\underline{y}$ and $\underline{z}$ ($\underline{x} = \lambda \underline{y} + (1-\lambda)\underline{z}$ with $\underline{y}, \underline{z} \neq \underline{x}$) where

(a) $\underline{y}$ and $\underline{z}$ are optimal solutions to LP
(b) if $x_i = 0$ then $y_i = z_i = 0$



<u>Claim 2</u> Let L be line through $\underline{y}$ and $\underline{z}$ and $\underline{x}$. Every vector $\underline{v}$ on L satisfies

(a) $\underline{c}^T\underline{v} = \underline{c}^T\underline{x}$
(b) $A\underline{v} = \underline{b}$
(c) If $x_i = 0$ then $v_i = 0$

<u>Claim 3</u> There is some $\underline{x}'$ on L such that $\underline{x}'$ is an optimal solution and has more zero entries than $\underline{x}$.

Claim 1 $\underline{x}$ can be written as a convex combination of $\underline{y}$ and $\underline{z}$ ($\underline{x} = \lambda \underline{y} + (1-\lambda)\underline{z}$ with $\underline{y}, \underline{z} \neq \underline{x}$) $\lambda \in [0,1]$ where

(a) $\underline{y}$ and $\underline{z}$ are optimal solutions to LP

(b) if $x_i = 0$ then $y_i = z_i = 0$

Pf Know $\underline{x}$ is not extreme point solution.

By defn of extreme, can write $\underline{x} = \lambda \underline{y} + (1-\lambda)\underline{z}$ (1)

for $\underline{y}, \underline{z}$ feasible and $\underline{x} \neq \underline{y}, \underline{z}$.

Know $\underline{y}, \underline{z}$ feasible, so to show $\underline{y}, \underline{z}$ optimal must show they are as good as $\underline{x}$

i.e. $\underline{c}^T\underline{x} = \underline{c}^T\underline{y} = \underline{c}^T\underline{z}$.

Since $\underline{x}$ is optimal, know that $\underline{c}^T\underline{y} \leq \underline{c}^T\underline{x}$, $\underline{c}^T\underline{z} \leq \underline{c}^T\underline{x}$ (2)

If $\underline{c}^T\underline{y} < \underline{c}^T\underline{x}$ then

$$\begin{aligned} \underline{c}^T\underline{x} = \underline{c}^T(\lambda \underline{y} + (1-\lambda)\underline{z}) &= \lambda \underline{c}^T\underline{y} + (1-\lambda)\underline{c}^T\underline{z} \\ &< \lambda \underline{c}^T\underline{x} + (1-\lambda)\underline{c}^T\underline{x} \\ &= \underline{c}^T\underline{x} \quad \text{a contradiction} \end{aligned}$$

So $\underline{c}^T\underline{y} = \underline{c}^T\underline{x}$. Similarly $\underline{c}^T\underline{z} = \underline{c}^T\underline{x}$. Shows (a)

For (b) Suppose $x_i = 0$

We know $\underline{y}, \underline{z}$ feasible so $y_i \geq 0$, $z_i \geq 0$

If $y_i > 0$ then

$$\begin{aligned} x_i &= \lambda y_i + (1-\lambda) z_i \\ &> \lambda \cdot 0 + (1-\lambda) 0 = 0 \end{aligned}$$

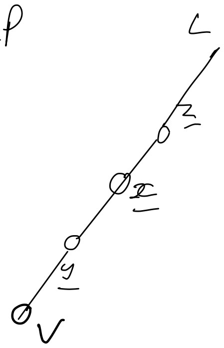
contradicting $x_i = 0$. Hence $y_i = z_i = 0$. Shows (b).

$\square$

Claim 1 $\underline{x}$ can be written as a convex combination of $\underline{y}$ and $\underline{z}$ ($\underline{x} = \lambda \underline{y} + (1-\lambda)\underline{z}$ with $\underline{y}, \underline{z} \neq \underline{x}$) where

(a) $\underline{y}$ and $\underline{z}$ are optimal solutions to LP

(b) if $x_i = 0$ then $y_i = z_i = 0$



Claim 2 Let L be line through $\underline{y}$ and $\underline{z}$ and $\underline{x}$. Every vector $\underline{v}$ on L satisfies

(a) $\underline{c}^T\underline{v} = \underline{c}^T\underline{x}$

(b) $A\underline{v} = \underline{b}$

(c) If $x_i = 0$ then $v_i = 0$

Pf (claim 2). If $\underline{v}$ on line L then $v = \theta \underline{y} + (1-\theta)\underline{z}$ with $\theta \in \mathbb{R}$

$$\begin{aligned} \underline{c}^T\underline{v} = \underline{c}^T(\theta\underline{y} + (1-\theta)\underline{z}) &= \theta\, \underline{c}^T\underline{y} + (1-\theta)\,\underline{c}^T\underline{z} \\ &= \theta\, \underline{c}^T\underline{x} + (1-\theta)\,\underline{c}^T\underline{x} \quad \text{by claim } 1(a). \\ &= \underline{c}^T\underline{x} \quad \text{proves } 2(a) \end{aligned}$$

$$\begin{aligned} A\underline{v} = A(\theta\underline{y} + (1-\theta)\underline{z}) &= \theta A\underline{y} + (1-\theta)A\underline{z} \\ &= \theta\underline{b} + (1-\theta)\underline{b} \quad (\underline{y}, \underline{z} \text{ feasible by } 1(a)) \\ &= \underline{b} \quad \text{proves } 2(b) \end{aligned}$$

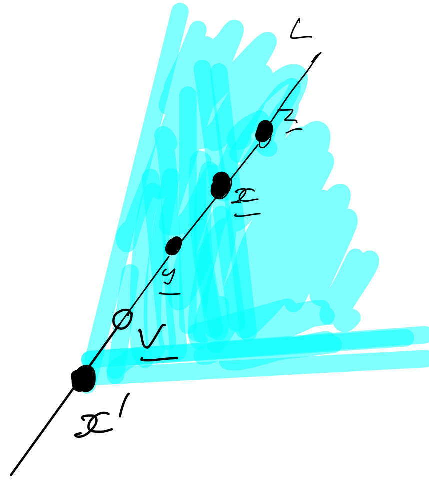
If $x_i = 0$ then $y_i = z_i = 0$ (by claim 1(b))

So
$$\begin{aligned} v_i &= \theta y_i + (1-\theta) z_i \\ &= \theta \cdot 0 + (1-\theta)\cdot 0 = 0. \end{aligned}$$

proves 2(c) $\square$

Claim 3 There is some $\underline{x}'$ on L such that $\underline{x}'$ is an optimal solution and has more zero entries than $\underline{x}$.

Pf start at $\underline{x}$ and move along L in a direction where at least one of the coordinates is decreasing.



Any zero entry of $\underline{x}$ stays zero as we move along L (claim 2(c))

Let $\underline{x}'$ be the first vector on L such that one of the positive entries becomes zero

Since $\underline{x} \geq \underline{0}$ the way we defined $\underline{x}'$ means $\underline{x}' \geq \underline{0}$

Also $A\underline{x}' = \underline{b}$ by claim 2(b). So $\underline{x}'$ feasible.

Also $\underline{c}^T\underline{x}' = \underline{c}^T\underline{x}$ by 2(a). So $\underline{x}'$ is optimal

By claim 2(c), $\underline{x}'$ has at least as many zero entries as $\underline{x}$ and in fact has at least one more by our choice of $\underline{x}'$.

$\square$

Pf of thm: Let $\underline{x}^*$ be the optimal solution of LP with most zero entries.

Then $\underline{x}^*$ is extreme point solution.

If not, by Claim 3, could find an optimal solution with more zero entries than $\underline{x}^*$, contradicting choice of $\underline{x}^*$.

$\square$

<u>Defn</u> Suppose we have an LP in standard equation form

$$\max \ \underline{c}^T \underline{x}$$
$$\text{sub to } \ A\underline{x} = \underline{b}, \quad \underline{x} \geq \underline{0}$$

We say $\underline{x}$ is a <u>basic feasible solution</u> of the LP if

(i) $\underline{x}$ is feasible

(ii) the columns of A corresponding to the non-zero entries of $\underline{x}$ should be linearly independence

<u>Example</u>

$$\max \ x_1 + x_2 + x_3$$
$$\text{Sub to } \ x_1 + \quad + x_3 = 1$$
$$x_2 + 2x_3 = 3$$
$$x_1, x_2, x_3 \geq 0$$

$$\underline{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad A = \begin{array}{c} \begin{array}{ccc} \underline{c}_1 & \underline{c}_2 & \underline{c}_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \end{array} \quad \underline{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

All solutions below are feasible (check). Which are basic feasible?

(a) $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ ✓ (b) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ✓ (c) $\begin{pmatrix} 1/2 \\ 2 \\ 1/2 \end{pmatrix}$ ✗

(a) $\underline{c_1}$ and $\underline{c_2}$ are linearly indep as they are standard basis vectors

(b) $\underline{c_2}$ and $\underline{c_3}$ are linearly indep (check)

(c) $\underline{c_1}, \underline{c_2}, \underline{c_3}$ are linearly dependent since $\underline{c_1} + 2\underline{c_2} - \underline{c_3} = \underline{0}$

Recap

Q If $A = \begin{pmatrix} \underline{c}_1 & \underline{c}_2 & \underline{c}_3 & \underline{c}_4 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ and $A\underline{v} = \underline{0}$

What can we say about linear dependence/independence of columns $\underline{c}_1, \underline{c}_2, \underline{c}_3, \underline{c}_4$ of A.

$$A\underline{v} = 2\underline{c}_1 + 3\underline{c}_3 = \underline{0}$$

$\underline{c}_1$ and $\underline{c}_3$ are linearly dependent.

Generally, if $A\underline{v} = \underline{0}$ then columns of A corresponding to non-zero entries of $\underline{v}$ are linearly dependent.

Thm For any LP in standard equation form

$$\max \ \underline{c}^T \underline{x}$$
$$\text{subj to } A\underline{x} = \underline{b}, \ \underline{x} \geq 0$$

$\underline{x}$ is a basic feasible solution (BFS) if and only if $\underline{x}$ is an extreme point solution (EPS)

(so BFS and EPS are the same but easier to check if a vector is a BFS).

Pf (By example) See printed notes for full proof.

Will show (i) not BFS $\Rightarrow$ not EPS
(ii) not EPS $\Rightarrow$ not BFS

Example from before

$$A = \begin{array}{c} \begin{array}{ccccc} \underline{c}_1 & \underline{c}_2 & \underline{c}_3 & \underline{c}_4 & \underline{c}_5 \end{array} \\ \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \underline{b} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} \quad \underline{y} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 4 \\ 0 \end{pmatrix}$$

(i) $\underline{y}$ is not BFS since $\underline{c}_1, \underline{c}_2, \underline{c}_3, \underline{c}_4$ are linearly dependent $[\underline{c}_1 - \underline{c}_2 + 2\underline{c}_3 - 2\underline{c}_4 = \underline{0}]$

$$A \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 0 \end{pmatrix} = \underline{0}$$

call this $\underline{r}$

Choose very small ε. Note $\underline{y} = \frac{1}{2}\underbrace{(\underline{y} + \varepsilon\underline{r})}_{\underline{y}_1} + \frac{1}{2}\underbrace{(\underline{y} - \varepsilon\underline{r})}_{\underline{y}_2}$

$\underline{y}_1$ is feasible since $A\underline{y}_1 = A(\underline{y} + \varepsilon\underline{r}) = A\underline{y} + \varepsilon A\underline{r}$
$= A\underline{y} = \underline{b}$

$\underline{y}_1 \geq \underline{0}$ because ε small. $\underline{y}_1$ is feasible.
Similarly $\underline{y}_2$ feasible.

So $\underline{y}$ is convex combination of $\underline{y}_1, \underline{y}_2$ both feasible. Hence $\underline{y}$ not an extreme point solution //

(ii) not proved in lectures

$\underline{w}$ is not extreme point solution since it is a convex combination $\underline{w} = \frac{1}{2}\underline{y} + \frac{1}{2}\underline{z}$ (see picture on next page) with $\underline{y}$ and $\underline{z}$ feasible, $\underline{w} \neq \underline{y}, \underline{z}$. (Aim: show $\underline{w}$ is not BFS)

so $A\underline{y} = \underline{b}$, $\underline{y} \geq \underline{0}$ and $A\underline{z} = \underline{b}$, $\underline{z} \geq \underline{0}$ (since $\underline{y}, \underline{z}$ feasible)

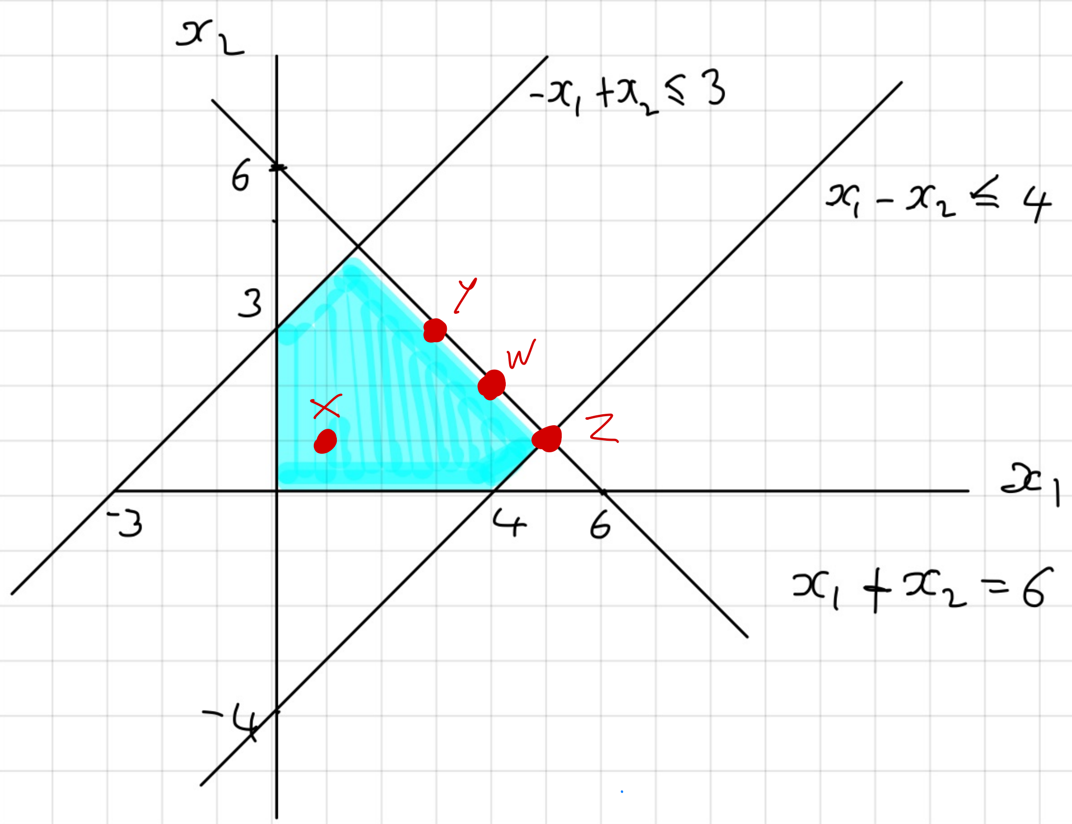
$$A(\underline{y} - \underline{z}) = A\underline{y} - A\underline{z} = \underline{b} - \underline{b} = \underline{0}.$$

Also $\underline{y} \neq \underline{z}$ (otherwise $\underline{w} = \underline{y} = \underline{z}$)
So $\underline{y} - \underline{z} \neq \underline{0}$

Columns of A corresponding to non-zero entries of $\underline{y} - \underline{z}$ are linearly dependent.
Also

$$(\underline{y} - \underline{z})_i \neq \underline{0} \Rightarrow y_i > 0 \text{ or } z_i > 0$$
$$\Rightarrow \tfrac{1}{2}y_i + \tfrac{1}{2}z_i > 0$$
$$\Rightarrow w_i > 0.$$

So columns of A corresponding to non-zero entries of $\underline{w}$ are linearly dependent.
So $\underline{w}$ is not BFS $\square$



<u>standard inequality form</u>

maximise $2x_1 + 3x_2$

sub to
$$-x_1 + x_2 \leq 3 \quad (1)$$
$$x_1 - x_2 \leq 4 \quad (2)$$
$$x_1 + x_2 \leq 6 \quad (3)$$
$$x_1, x_2 \geq 0$$

<u>standard equation form</u>

maximise $2x_1 + 3x_2$

sub to
$$\begin{aligned} -x_1 + x_2 + s_1 \quad\quad &= 3 \\ x_1 - x_2 \quad + s_2 \quad &= 4 \\ x_1 + x_2 \quad\quad + s_3 &= 6 \end{aligned}$$
$$x_1, x_2, s_1, s_2 \geq 0.$$

$$\begin{array}{cccc} & X & Y & Z \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 3 \\ 3 \end{pmatrix} & \begin{pmatrix} 5 \\ 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cccc} & X & Y & Z \\ \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = & \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 4 \end{pmatrix} & \begin{pmatrix} 3 \\ 3 \\ 3 \\ 4 \\ 0 \end{pmatrix} & \begin{pmatrix} 5 \\ 1 \\ 7 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$