Practice Set 4 Solutions

1. An investor is considering entering a business venture which involves a 50-50 chance of an income of £900 or £400 depending on the state of the economy. The initial investment involves a payment of I_0 =£650. Investor's initial wealth is $w_0 = £1{,}000$.

   a) Using the definition of a "fair gamble", explain whether this business venture is a fair gamble for the investor.

   Answer: A gamble is fair if the expected wealth is equal with the initial wealth.

   In this case:

   $E(w) = -I_0 + \frac{1}{2} \times (w_0 + 900) + \frac{1}{2} \times (w_0 + 400) = -I_0 + w_0 + 650 = w_0$ as the initial investment is 650.

   Alternative answer:

   The random gain/loss that the investor faces in this venture is $X = \{900 - 650, 400 - 650\} = \{250, -250\}$ with probabilities: $\{\frac{1}{2}, \frac{1}{2}\}$.

   $$E(w) = \frac{1}{2} \times (w_0 + 250) + \frac{1}{2} \times (w_0 - 250) = w_0$$

   Hence, this business venture is a fair gamble.

   b) Assume that the investor's utility function of wealth is $v(w) = \sqrt{w}$, with $w > 0$. Explain, giving reasons, whether the investor would invest I_0 or any other amount in this venture.

   Answer:

   It is sufficient to state the following:
   This investor is risk averse as $v''(w) = -\frac{1}{4} w^{-\frac{3}{2}} < 0$ for any $w > 0$. Hence, he is not willing to take on a fair gamble.

   Alternatively: in this particular case the expected utility of the gamble:

   $E\big(v(w)\big) = \frac{1}{2} \times \sqrt{1{,}000 + 900 - 650} + \frac{1}{2} \times \sqrt{1{,}000 + 400 - 650} = 31.37$

   Note that the initial investment needs to be paid in both state of the economy.

   Hence, the random gain that the investor faces is $X = \{900 - 650, 400 - 650\} = \{250, -250\}$ with probabilities: $\{\frac{1}{2}, \frac{1}{2}\}$.

Seeing that $U(w_0) = \sqrt{1{,}000} = 31.62$ we can conclude that $E\big(v(w)\big) < U(w_0)$.

Hence he is not willing to enter the business.

However, the investor can find the maximum amount he is willing to pay for the venture by looking at indifference condition between taking the gamble or not, taking into account his utility of wealth.

$E\big(v(w)\big) = U(w_0)$, or

$$E\big(v(w)\big) = \tfrac{1}{2} \times \sqrt{1{,}900 - I_0} + \tfrac{1}{2} \times \sqrt{1{,}400 - I_0} = \sqrt{1{,}000} = U(w_0)$$

With solution $I_0 \cong 634$

An alternative method: The maximum amount the investor is willing to pay in order to avoid this gamble when the investment is £650 is found by the indifference condition between taking the gamble and not taking the:

$E\big(v(w_0 + X)\big) = U(w_0 - c_x)$ where $X = \{250, -250\}$ with probabilities: $\{\frac{1}{2}, \frac{1}{2}\}$.

$$984.1229 = 1000 - c_x$$

$$c_x = 1000 - 982.1229 = 15.8771$$

Hence, the maximum sure amount he is willing to pay to not take the gamble is 15.878 This means the maximum investment he is willing to commit to pay for the venture is approximately the initial investment £650 less the certainty equivalent of the gamble. Hence $I_0 = 650 - 15.878 = 634.12$

c) Suppose the investor has instead a utility function described by $v(y) = ay$ where $a > 0$. What is the amount the investor is prepared to invest to take part in the venture?

Answer:

It is sufficient to state: This investor is risk neutral as his utility of wealth is linear. Hence, he is indifferent to weather to accept the fair gamble or not.

Alternatively: in this particular case:

$$E\big(v(w)\big) = \tfrac{a}{2} \times (-650 + 1{,}000 + 900) + \tfrac{a}{2} \times (-650 + 1{,}000 + 400) = 1{,}000a$$

Seeing that $U(w_0) = 1{,}000a$ we can conclude that $E\big(v(w)\big) = U(w_0)$. And hence, he is indifferent whether to take the gamble or not

The exact amount the investor is willing to pay in order to enter the gamble is I_0 and it is found by:

$$E\big(v(w_0 + X)\big) = U(w_0)$$

$$\frac{a}{2} \times (1{,}000 + 900 - I_0) + \frac{a}{2} \times (1{,}000 + 400 - I_0) = 1{,}000a$$

$$I_0 = 650$$

Answers that state without calculations that the amount the investor is willing to pay is exactly 650, since this is what it makes the business venture a fair gamble are accepted as well.

d) If the investor's utility function is $v(w) = w^2$, is he willing to invest the required initial investment of £650?
Answer:

This investor is risk seeking as $v''(w) = 2 > 0$ for any $w > 0$. Hence, he is always willing to take on a fair gamble.

Alternative answer: in this particular case the expected utility of the gamble:

$E\big(v(w)\big) = \frac{1}{2} \times (1{,}000 + 900 - 650)^2 + \frac{1}{2} \times (1{,}000 + 400 - 650)^2 =$
$1{,}062{,}500 > 1{,}000{,}000 = U(1{,}000)$

The investor definitely will enter the gamble at £650 investment. The investor would in fact would be willing to in fact invest even more than £650 as he attaches a higher utility to an incremental increase in wealth to an incremental decrease as $v''(w) > 0$

e) Explain the reason for the different investment values (under which the investor is willing to invest) in the situations described in points b), d), and e).

Answer

When evaluating an investment, an investor compares the expected utility of wealth from taking a gamble with the sure utility of keeping the initial wealth and not entering the gamble: $E\big(v(w)\big) = U(w_0)$

Each investor has different preferences for risk and this is reflected in their utility of wealth functions and not in the wealth itself.

Risk averse investors have their utility of wealth concave and they attach a lower utility to an incremental increase in wealth to an incremental decrease. This is due to the fact that they have decreasing marginal utility of wealth. Risk seeking investors have convex utilities of wealth which imply increasing marginal utility of wealth. Hence, they attach a higher utility to an incremental increase in wealth versus to an incremental decrease.

Risk neutral investors have linear utility functions with constant marginal utility of wealth and they are indifferent about accepting or rejecting a fair bet.

2. An investor has the choice between two assets, A and B.
The annual return on asset A is a continuous random variable uniformly distributed on 3% and 7%.
Note that the probability density function for a continuous uniform distribution on the interval $[a, b]$ is $f(x) = \frac{1}{b-a}$for $a \leq x \leq b$ and $f(x) = 0$ otherwise.
The annual return on investment B will only take discrete values with probabilities given in the following table:

Probability	0.3	0.3	0.2	0.2
Return	4%	-4%	10%	-8%

For each investment calculate the following statistics:

i) Expected annual return

Answer:

$$E_A \equiv E_A(X) = \int_{0.03}^{0.07} \frac{x}{0.04} dx = \left[\frac{x^2}{0.08}\right]_{0.03}^{0.07} = 0.05$$

Or

$$E_A \equiv E_A(X) = \frac{0.03 + 0.07}{2} = 0.05$$

$$E_B \equiv E_B(X) = 0.3 \times 0.04 - 0.3 \times 0.04 + 0.2 \times 0.10 - 0.2 \times 0.08 = 0.2 \times 0.02 = 0.004$$

ii) Variance of annual return

Answer:

$$V_A \equiv Var_A(X) = \int_{0.03}^{0.07} (x - 0.05)^2 \frac{1}{0.04} dx = 0.000133$$

Or

$$V_A \equiv Var_A(X) = \frac{(0.07 - 0.03)^2}{12} = 0.000133$$

$$V_B \equiv Var_B(X) = 0.3 \times (0.04 - 0.004)^2 + 0.3 \times (-0.04 - 0.004)^2 + 0.2 \times (0.10 - 0.004)^2 + 0.2 \times (-0.08 - 0.004)^2 = 0.004224$$

iii) Semi-variance of annual return

Answer:

$$SV_A \equiv \int_{0.03}^{0.05} (x - 0.05)^2 \frac{1}{0.04} dx = 0.000067$$

Note: for B the returns need first to be ordered in ascending order: -8%, -4%, 8%, 10% with their respective probabilities. Hence:

$$SV_B = 0.2 \times (-0.08 - 0.004)^2 + 0.3 \times (-0.04 - 0.004)^2 = 0.001992$$

iv) Shortfall probability with a required level of annual return of 6%
Answer:

$$SP_A(X \leq 0.06) = Pr_A(X \leq 0.06) = \int_{0.03}^{0.06} \frac{1}{0.04} dx = \frac{3}{4} = 75\%$$

$$SP_B(X \leq 0.06) = 0.3 + 0.3 + 0.2 = 0.8 = 80\%$$

v) Explain which investment you would choose based on your answers to parts a) to d).
Answer:
Asset A has higher return and lower variance and semi variance of return than asset B. Also the shortfall probability for a required level of return of 6% for asset A is 75% while for asset B is 80%. Based on all statistics Asset A performs better than B.

3. There are 100 possible states for the market over the next year with each state being equally likely. Two assets, *A* and *B* have returns over the next year of R_A and R_B respectively. These returns depend on the state of the market and are given below as percentages:

Market condition	R_A	R_B
1	-5.0	-10.0
2	-3.5	-7.5
3	-3.0	-6.5
4	-2.8	-5.0
5	-2.5	-2.5
6	-2.0	-2.0
...	...	...
98	6.5	6.5
99	9.0	9.0
100	10.0	10.0

For *X* = *A*,*B*: calculate:

(i) $SP(R_A;\ -2.5) = 0.04$ as probability of each state is 0.01
$SP(R_B;\ -2.5) = 0.04$ by the same argument

(ii) $VaR(R_X; q = 5\%)$ and $VaR(R_X; q = 1\%)$

$VaR(R_A; q = 5\%) = -L$ where $L = max\{R_i \colon \Pr(R_A < R_i) \leq 0.05\}$

Looking at the table $\Pr(R_A < -2) = 0.05$

$VaR(R_A; q = 5\%) = 2$

Similarly, $VaR(R_B; q = 1\%) = -L$ where $L = max\{R_i \colon \Pr(R_B < R_i) \leq 0.01\}$

$VaR(R_B; q = 5\%) = 2$

iii) $VaR(R_A; q = 1\%) = -L$ where $L = max\{R_i: \Pr(R_A < R_i) \leq 0.01\}$

Looking at the table $\Pr(R_A < -3.5) = 0.01$

$VaR(R_A; q = 1\%) = 3.5$

$VaR(R_B; q = 1\%) = -L$ where $L = max\{R_i: \Pr(R_B < R_i) \leq 0.01\}$

Looking at the table $\Pr(R_B < -7.5) = 0.01$
$VaR(R_B; q = 1\%) = 7.5$