Week 5

Def of dividend Slide 27

8.1 → Q1 Different types of dividends

Types of dividends

① Discrete <u>absolute</u> dividends

times	0	t_1	t_2	$\cdots$	t_n	T
Amount paid		D_1	D_2	$\cdots$	D_n	
		$d_1 S(t_1)$	$d_2 S(t_2)$		$d_n S(t_n)$	

② Discrete <u>proportional</u> dividends → i

③ <u>Continuous</u> <u>proportional</u> dividends → r

t $\quad$ $t+\Delta t$

If $\Delta t > 0$ is a small time interval

then the amount paid from t to $t+\Delta t$ is $q S(t+\Delta t)\Delta t$

Assumptions:

① Dividends can be paid either in cash or shares

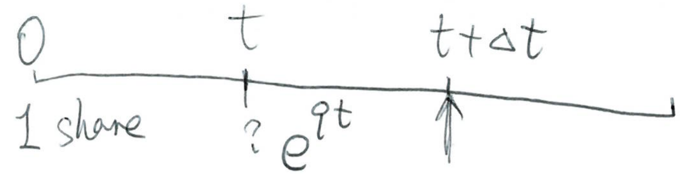
~~Assum~~

② A dividend will be re-invested in the underlying share

8.2 Continuous dividend rates   Type 3

Lemma 8.1: how many shares do we own?

Suppose that the dividend is paid continously and is reinvested in the share.



dividend: $q \cdot S(t+\Delta t) \cdot \Delta t$ $\Delta t \to 0$

Stock price: $S(t+\Delta t)$

$N(t)$: the number of sharers at time t

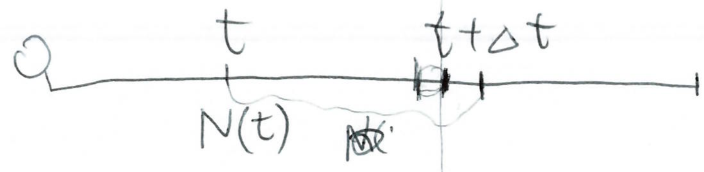
$N(0) = 1$

$N(t) = e^{qt}$

r e^{rt}

Def

Dividend: $q\,N(t)\;S(t+\Delta t)\;\Delta t$



number to be reinvested: $\dfrac{\text{Dividend}}{\text{price}}$

$$= \frac{q\,N(t)\,S(t+\Delta t)\cdot\Delta t}{S(t+\Delta t)}$$

$$= q\,N(t)\,\Delta t$$

before paying dividend	after paying dividend
$N(t+\Delta t)$	$N(t+\Delta t) + qN(t)\Delta t$
$\approx N(t)$	$\approx N(t) + qN(t)\Delta t$

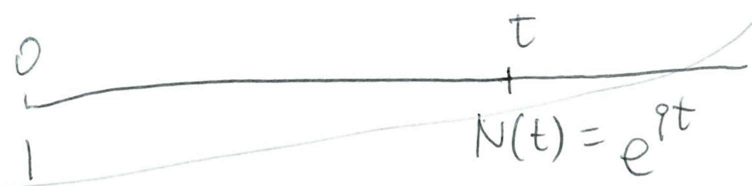
reinvestment

$$\boxed{\frac{N(t+\Delta t)-N(t)}{\Delta t} = qN(t)} \Rightarrow N'(t) = qN(t)$$

Number of shares at $t+\Delta t$:

$$N(t+\Delta t) = N(t) + qN(t)\Delta t$$

Def



Q2: price when dividends are involved

q continous dividend

r interest rate compounded continuously

price C $R(S(t))$ payoff function

Th 5.2 $C = e^{-rt} \tilde{E}(R(S(t)))$

Lemma 8.2

share q reinvested continuously

$S(0)$

$M(t)$: cost at time $t>0$ of the portfolio which at time $t=0$ consist of 1 share

r.

Then $\boxed{\tilde{E}(M(t)) = S(0)e^{rt}}$,

$\tilde{E}$: RNP

$\boxed{\text{1 share}}$

$$C = e^{-rt}\tilde{E}(\ S(t)\ )$$

0 — t — T

Price: $S(0)$ $\quad \tilde{E}\left[\boxed{S(t) = S(0)e^{rt}}\right]$

$$S(0) = e^{-rt}\tilde{E}(S(t))\ \cancel{R(t)=0}$$

$$R(S(t)) = S(t)$$

Cost/payoff $M(0)$ $\quad \tilde{E}\left[\boxed{M(t)}\right]$ = num of shares × price of the share

$\tilde{E}$

At time t: $S(t) = S(0)e^{rt}$

$$M(t) = N(t)\cdot S(t)$$

$$\overset{\text{Lemma 8.1}}{=} \boxed{e^{qt}\cdot S(t)}$$

$M(t) = S(t)$ because in the portforlio we only have 1 share at ~~the~~ time 0 $\Leftrightarrow$ M portfolio is equivalent to 1 share

~~WS④~~

$$\tilde{E}(M(t) - S(0)e^{rt}) = 0$$

$$\tilde{E}(M(t)) = S(0)e^{rt} \quad ▨$$

For exam: no proof

Apply $\tilde{E}(\boxed{M(t)}) = S(0)e^{rt}$

$S(t)$

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Corollary 8.1

Suppose all the conditions of Lemma 8.2 are satisfied, Then

$$\tilde{E}\left(\boxed{S(t)}\right) = S(0)\, e^{(r-q)t}$$

Proof: $\tilde{E}\left(S(0)e^{rt}\right) = \tilde{E}\left(e^{qt} S(t)\right)$

$$S(0)\, e^{rt} = \tilde{E}\left(e^{qt} S(t)\right)$$

$$\tilde{E}\left(S(t)\right) = S(0)\, e^{rt} \cdot e^{-qt}$$

Theorem 8.1

Suppose

(a) $S(t) = S e^{\mu t + \sigma W(t)}$ GBM

r interest compounded

(b) Dividend q continously paid reinvested

Then $\tilde{S}(t) = S\, e^{\tilde{\mu} t + \sigma W(t)}$, $\tilde{\mu} = r - q - \frac{\sigma^2}{2}$ RNP

Corollary 8.2

Suppose the conditions of Th 8.1 are satisfied.

Derivative $R(S(t))$ ← payoff

Then ~~$\tilde{E}(R(t)$~~ $\tilde{E}\Big(R(S(t))\Big) = E\Big(R(\tilde{S}(t))\Big)$

price of the derivative

$$C = e^{-rT} E\Big(R(\tilde{S}(t))\Big) = e^{-rt}\int_{-\infty}^{\infty} R\left(Se^{\tilde{\mu}t + \sigma\sqrt{t}x}\right) f(x)\,dx$$

$f(x)$: standard normal density : $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

In particular, if $R(\tilde{S}(T)) = (S(T) - K)^+$ Call (K, T)

$$C = e^{-rT} E\left(\tilde{S}(T) - K\right) = e^{-rT}\int_{-\infty}^{\infty}\left(Se^{\tilde{\mu}T + \sigma\sqrt{T}x} - k\right)^+ f(x)\,dx$$

quick proof of Th 8.1

× Part 1: The risk-neutral process has the form $\underline{\tilde{S}(t) = S e^{\tilde{\mu} t + \sigma W_t}}$

↓

✓ Part 2: $\tilde{\mu} = r - q - \frac{\sigma^2}{2}$

$$\tilde{E}(S(t)) \overset{C8.2}{=} E(\tilde{S}(t)) \overset{\text{Part 1}}{=} E\left(S e^{\tilde{\mu} t + \sigma W_t}\right) = S e^{\tilde{\mu} t} \underline{E(e^{\sigma W_t})}$$

$$E(e^{\sigma W_t}) \overset{\text{Week 1}}{=} e^{\frac{\sigma^2}{2} t}$$

$$\boxed{\tilde{E}(S(t))} = S e^{\tilde{\mu} t} e^{\frac{\sigma^2}{2} t} = \boxed{S e^{(\tilde{\mu} + \frac{\sigma^2}{2}) t}}$$

By Corollary 8.1: $\boxed{\tilde{E}(S(t))} = \boxed{S(0) e^{(r-q)t}}$ $\qquad S = S(0)$

$$S e^{\boxed{\tilde{\mu} + \frac{\sigma^2}{2}} t} = S e^{\boxed{(r-q)} t}$$

$$\tilde{\mu} + \frac{\sigma^2}{2} = r - q \qquad \boxed{\tilde{\mu} = r - q - \frac{\sigma^2}{2}}$$

8.4 Application of Th 8.1

Suppose $S(t) = s e^{\mu t + \sigma W_t}$. GBM

~~Div~~ $\boxed{\text{Dividend} \quad q}$, r.

Price of Call (K, t) ?

With dividend

$C_q(S, t, K, \sigma, r)$

Without dividend

$C(S, t, K, \sigma, r)$

Theorem 8.2

$$C_q(S, t, K, \sigma, r) = C(e^{-qt} S, t, K, \sigma, r)$$

Proof: For both C_q and C, the payoff function is

$\cancel{R((S(t) - K)^+)}$

$R(S(t)) = (S(t) - K)^+$ ← payoff for European call option

By Th 5.2 and 5.3

$$C_q(S, t, K, \sigma, r) \overset{\text{Th5.2}}{=} e^{-rt} \tilde{E}\left[(S(t) - K)^+\right]$$

By Th 8.1

$$C_q(S, t, K, \sigma, r) \overset{C8.2}{=} e^{-rt} E\left[(\tilde{S}(t) - K)^+\right]$$

$$\overset{\text{Th 8.1}}{=} e^{-rt} E\left[\left(S e^{\overbrace{(r-q-\frac{\sigma^2}{2})}^{\tilde{\mu}}t + \sigma W_t} - K\right)^+\right]$$

$$= \boxed{e^{-rt} E\left[\left(\boxed{(Se^{-qt})} e^{(r-\frac{\sigma^2}{2})t + \sigma W_t} - K\right)^+\right]}$$

If $S(t) = \bar{S} e^{\mu t + \sigma W_t}$ GBM and no dividend is paid

then $C \equiv C(\bar{S}, t, K, \sigma, r) = \boxed{e^{-rt} E\left[\left(\boxed{\bar{S}} e^{(r-\frac{\sigma^2}{2})t + \sigma W_t} - K\right)^+\right]}$

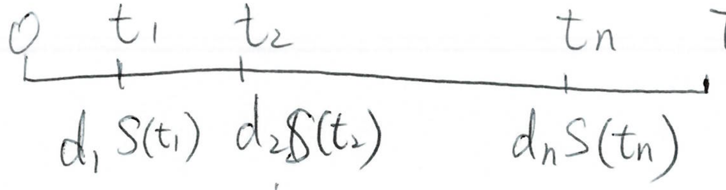
If $\bar{S} = e^{-qt} S_0$, then $C_q(S, t, K, \sigma, r) = C(e^{-qt}S, t, K, \sigma, r)$ ///

WS⑨

8.5 Discrete proportional dividends

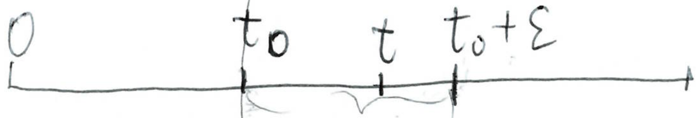
Type 2 dividends

$0 < d_j < 1$ for $\forall j,\ 1 \le j \le n$



Q: What happens to the price of the share shortly after t_0

$n = 1$



$dS(t_0)$ reinvested immediately after time t_0

$0 < d < 1$

$$\lim_{\varepsilon > 0,\ \varepsilon \to 0} S(t_0+\varepsilon) = S(t_0) - \underline{dS(t_0)} = (1-d)\,S(t_0)$$

$S(t_0+\varepsilon) = (1-d)S(t_0)$

Proof: If $\boxed{S(t) < (1-d)\,S(t_0)}$ $(*)$ for all $t \in [t_0, t_0+\varepsilon)$

Short-sell a share: $t=0$: Borrow a share from a bank, and sell it (can use money)

$t=T$ return the share together with dividend the share would have paid to the owner

Arbitrage opportunity:

	0	t_0	$\bar{t}$	$t_0+\varepsilon$
CF		$S(t_0)$	$-S(\bar{t})-dS(t_0)$	

1. short-sell the share at time $t=t_0$ for $S(t_0)$

2. At time $\bar{t}$, $t_0<\bar{t}<t_0+\varepsilon$

   buy the share for $S(\bar{t})$ and return it to the owner of share together with the dividend $dS(t_0)$

   $S(\bar{t}) \overset{\varepsilon\to 0}{\approx} S(t_0)$

   $\varepsilon \to 0$

Return: $S(t_0)-S(\bar{t})-dS(t_0)$ ignore interest $\varepsilon\to 0$

$= (1-d)S(t_0)-S(\bar{t})$ $S(\bar{t})<(1-d)S(t_0)$

$\overset{*}{\geq} (1-d)S(t_0)-(1-d)S(t_0)=0$ $t_0\approx\bar{t}\approx t_0+\varepsilon$ $\varepsilon\to 0$

Positive return $\longrightarrow$ arbitrage

$N'(t) = q N(t)$

$$\begin{cases} N(t) = A e^{qt} \\ N(0) = 1 \end{cases}$$, where A is an constant

$A = 1$

$N(t) = e^{qt}$ ☒

$\cancel{S(t_0^+) = (1-d)\, S(t_0)}$

$$\lim_{\varepsilon \to 0} S(t_0+\varepsilon) = (1-d)\, S(t_0)$$

intuition: $S(t_0+\varepsilon) \neq S(t_0)$

d

Summary of important conclusions of Discrete proportional dividends

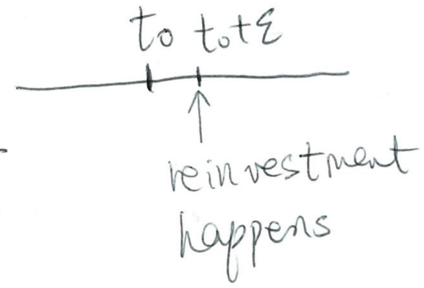
1. $$\lim_{t \geq t_0,\ t \to t_0} S(t) = (1-d)\, S(t_0)$$

2. Reinvest of the dividend: buy $\frac{d}{1-d}$ additional shares

price: $S(t_0+\varepsilon) = (1-d)\, S(t_0)$

Amound of dividend: $d\, S(t_0)$

$$\text{number of share} = \frac{\text{dividend}}{\text{price}} = \frac{d\, S(t_0)}{(1-d)\, S(t_0)} = \frac{d}{1-d}$$



3. Portfolio: 1 share

The value of the portfolio is $S(t)$ at $t \leq t_0$

$\frac{1}{1-d} S(t) \qquad t > t_0$

when $t \le t_0$: 1 share cost $S(t) \times 1 = S(t)$

$t > t_0$: $1 + \frac{d}{1-d} = \frac{1}{1-d} \qquad S(t) \times \frac{1}{1-d} = \frac{1}{1-d} S(t)$

Theorem 8.3 Additional assumption GBM, RNP

Suppose that:

1. The price of an asset follows the GBM, $S(t) = S e^{\mu t + \sigma W(t)}$
2. Discrete proportional dividend paid at time t_0 at a rate d, $0 < d < 1$
3. r

Then the risk-neutral process is

$$\tilde{S}_1(t) = \begin{cases} \tilde{S}(t), & \text{if } t \le t_0 \\ (1-d)\tilde{S}(t) & \text{if } t > t_0 \end{cases}$$

where $\tilde{S}(t) = S e^{\tilde{\mu} t + \sigma W(t)}$, and $\tilde{\mu} = r - \frac{\sigma^2}{2}$.

$\tilde{E}(R(S(t))) = E(R(\tilde{S}_1(t)))$