Tutorial 4

(6)

$$m_0 \bullet \quad \Rightarrow \quad \overset{u}{\longleftarrow} m_0/2 \qquad \overset{h\nu}{\rightsquigarrow}$$

4-momenta before and after:

Before: $\bar{P}_0 = m_0 (1, 0, 0, 0)$

After: $\bar{P}_1 = \frac{m_0}{2} \gamma(u) (1, -u, 0, 0)$

$$\bar{P}_\gamma = (h\nu, h\nu, 0, 0)$$

Conservation of 4-momentum:

$$\bar{P}_0 = \bar{P}_1 + \bar{P}_\gamma$$

The t and x components of the conservation equation give:

$$m_0 = \frac{m_0}{2} \gamma(u) + h\nu$$

$$0 = -\frac{m_0}{2} \gamma(u) u + h\nu$$

Subtracting the two equations gives:

$$1 = \frac{1}{2} \gamma(u) (1+u)$$

$$= \frac{1}{2} \sqrt{\frac{1+u}{1-u}} \quad \Rightarrow \quad u = \frac{3}{5}$$

This gives $\gamma(u) = \frac{1}{\sqrt{1-u^2}} = \frac{5}{4}$

Using this result in the first equation gives:

$$m_0 = \frac{m_0}{2}\,\frac{5}{4} + h\nu$$

$$= \frac{5}{8} m_0 + h\nu$$

$$\Rightarrow \quad h\nu = \frac{3}{8} m_0$$

(10)

a) $\xrightarrow{v} \qquad \xleftarrow{v}$

The incoming 4-momenta are

$$P_1 = m_p \gamma_v (1, v, 0, 0)$$

$$P_2 = m_p \gamma_v (1, -v, 0, 0)$$

$$P_3 = M_H (1, 0, 0, 0)$$

Conservation of energy implies:

$$2 m_p \gamma_v = M_H = 125 \, m_p$$

$$\rightarrow \gamma_v = \frac{125}{2} = \frac{1}{\sqrt{1-v^2}} \rightarrow 1 - v^2 = \left(\frac{2}{125}\right)^2$$

$$v = \sqrt{1 - \left(\frac{2}{125}\right)^2} = 0.99987$$

b) Before: $P_1 = m_p \gamma_v (1, v\cos\theta_1, -v\sin\theta_1, 0)$

$P_2 = m_p \gamma_v (1, v\cos\theta_2, +v\sin\theta_2, 0)$

After: $P_3 = M_H \gamma_u (1, u, 0, 0)$

Conservation of momentum along the y axis implies $\theta_1 = \theta_2$.

Conservation of momentum along the x axis implies:

$$2 m_p \gamma_v \cos\theta_1 = M_H \gamma_u u$$

$$2 \gamma_v \cos\theta = 125 \gamma_u u$$

$$\cos\theta_1 = \frac{125 \gamma_u u}{2 \gamma_v} = u = \frac{\sqrt{3}}{2} \rightarrow \theta = \frac{\pi}{6} = 30^\circ$$

Conservation of energy: $2 m_p \gamma_v = M_H \gamma_u = 125 m_p \gamma_u$

$$\rightarrow \quad 2 \gamma_v = 125 \gamma_u$$

$$\Rightarrow v = \frac{6\sqrt{434}}{125} = 0.999968 \ldots$$

(11)

a) $[X, Y](f) = X(Y(f)) - Y(X(f))$

$$= X^b \partial_b (Y^a \partial_a f) - Y^b \partial_b (X^a \partial_a f)$$

$$= [X^b (\partial_b Y^a) - Y^b (\partial_b X^a)] \partial_a f$$

$$+ \underbrace{(X^b Y^a - Y^b X^a)}_{\text{anti-symmetric } a \leftrightarrow b} \underbrace{\partial_a \partial_b}_{\text{symmetric } a \leftrightarrow b} f$$

$$= [X, Y]^b \partial_b f$$

from which we identify

$$[X, Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a$$

b) · Direct calculation

$$[X, Y]^{a'} = X^{b'} \partial_{b'} Y^{a'} - (X \leftrightarrow Y)$$

$$= \frac{\partial x^{b'}}{\partial x^b} X^b \frac{\partial x^c}{\partial x^{b'}} \partial_c \left(\frac{\partial x^{a'}}{\partial x^a} Y^a \right) - (X \leftrightarrow Y)$$

$$= \frac{\partial x^{a'}}{\partial x^a} X^c \partial_c Y^a + \cancel{X^c Y^a \frac{\partial^2 x^{a'}}{\partial x^c \partial x^a}}$$

$$- \left(\frac{\partial x^{a'}}{\partial x^a} Y^c \partial_c X^a + \cancel{Y^c X^a \frac{\partial^2 x^{a'}}{\partial x^c \partial x^a}} \right)$$

$$= \frac{\partial x^{a'}}{\partial x^a} [X, Y]^a$$

- Alternative:

Note that

$$[X, Y]^a = X^b \nabla_b Y^a - Y^b \nabla_b X^a$$

since the terms with the Christoffels cancel. Then, since $[X, Y]$ can be written in a manifestly covariant way it will transform as a vector.