MTH6105 – Algorithmic Graph Theory **Spring 2024**

Problem Sheet 3 **F. Fischer**

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

---

1. Show that a graph G is a tree if and only if it contains no loops and a unique u–v-path for every $u, v \in V(G)$.

   (a) For the direction from right to left, consider a graph G without loops and with unique u–v-paths, and show that G is connected and acyclic.

   (b) For the direction from left to right, show that if T is a tree, then it contains unique u–v-paths.

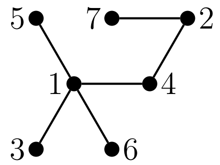
   Solution:

   (a) Assume that G contains no loops and a unique u–v-path for every $u, v \in V$. Observe that this means that G is connected, and assume for contradiction that G contains a cycle $v_1 e_1 v_2 e_2 \dots v_m e_m v_1$. Then $v_1 e_1 v_2 e_2 \dots v_m$ and $v_1 e_m v_m$ are distinct v_1–v_m-paths in G, contradicting the assumption that such paths are unique. Thus G is connected and does not contain any cycles, i.e., it is a tree.

   (b) Assume that G is a tree, and consider $u, v \in V(G)$. By definition G is connected, so it contains at least one u–v-path. Assume for contradiction that it contains two distinct u–v-paths $s_0 s_1 s_2 \dots s_m$ and $t_0 t_1 t_2 \dots t_\ell$. Let $i \in \{0, \dots, m-2\}$ be the smallest value such that $s_{i+1} \neq t_{i+1}$, and $j \in \{i+1, \dots, m\}$ the smallest value such that $s_j \in \{t_{i+1}, t_{i+2}, \dots, t_\ell\}$. Then v_{i+1} is the first vertex where the first path diverges from the second path, and v_j the first vertex thereafter where the two paths converge again; such vertices must exist because the paths are distinct but both start at u and end at v. Let $k \in \{i+1, i+2, \dots, \ell\}$ such that $s_j = t_k$. Then $s_i s_{i+1} \dots s_j t_{k-1} t_{k-2} \dots t_{i+1} t_i$ is a cycle in G, which contradicts the assumption that G is a tree.

---

2. (a) Give the Prüfer code of the following tree.

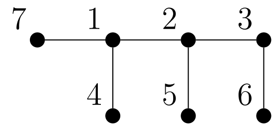


   (b) Draw the tree with Prüfer code $(1, 2, 3, 2, 1)$

Solution:

(a) The tree has Prüfer code $1, 1, 1, 4, 2$.

(b) The tree looks as follows.



---

3. For $n \in \{0, 1, 2, 3, \dots\}$, let Q_n be the simple graph with

$$V(Q_n) = \{X : X \subseteq [n]\},$$
$$E(Q_n) = \{XY : X, Y \in V(Q_n),\ |(X \setminus Y) \cup (Y \setminus X)| = 1\}.$$

(a) Draw Q_0, Q_1, Q_2, and Q_3.

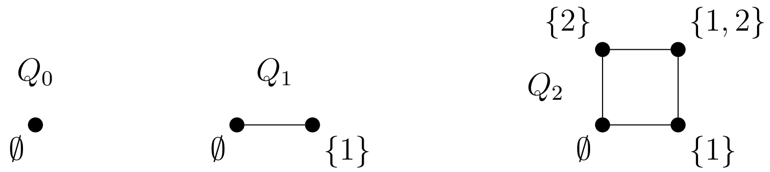
(b) Determine $d_{Q_{13}}(\{1, 3\})$.

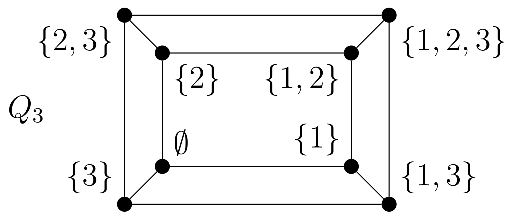
(c) Give all values of n for which Q_n is a tree. Justify your answer.

(d) Show that Q_n is connected for all n. You may want to consider $X, Y \in V(Q_n)$ such that $|(X \setminus Y) \cup (Y \setminus X)| = k$, and show existence of an X–Y-path by induction on k.

Solution:

(a) The graphs look as follows.





(b)
$$\begin{aligned} d_{Q_{13}}(\{1, 3\}) &= |N_{Q_{13}}(\{1, 3\})| \\ &= |\{\{1, 3\} \setminus \{x\} : x \in \{1, 3\}\} \cup \{\{1, 3\} \cup \{x\} : x \in [13] \setminus \{1, 3\}\}| \\ &= 13 \end{aligned}$$

(c) As we can see above, Q_0 and Q_1 are connected and acyclic, and thus trees. If $n \geq 2$, Q_n contains the cycle $\emptyset, \{1\}, \{1, 2\}, \{2\}, \emptyset$ and is therefore not a tree.

(d) Let $n \in \{0, 1, 2, 3, \dots\}$. For $X, Y \in V(Q_n)$, let $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$ denote the symmetric difference of X and Y. Consider $X, Y \in V(Q_n)$, and let $k = |X \triangle Y|$. We prove by induction on k that there exists an X–Y-walk in Q_n. If $k = 1$, then $XY \in E(Q_n)$, so there exists an X–Y-walk in Q_n. Now assume that $k \geq 2$. Let $z \in X \triangle Y$ and $Z = X \triangle \{z)$. Then $Z \in E(Q_n)$. Moreover, $|X \triangle Z| = 1$ and $|Z \triangle Y| = k - 1$, so by the induction hypothesis exists an X–Z-walk and a Z–Y-walk in Q_n. These two walks can be combined into an X–Y-walk. Since X and Y were chosen arbitrarily, Q_n is connected.