Week 4

Slide 33

Put (K, T)

1. If $S(T) \geq K$

0		T
Borrow from the bank	C	$-Ce^{rT}$
to buy the option	$-C$	0

2. If $S(T) < K$

0		T
Borrow from the bank	C	$-Ce^{rT}$
to buy the option	$-C$	
		buy the share $-S(T)$
		sell the share at K

$$-Ce^{rT} + (K - S(T))^{+} = 0$$

~~$C = e^{-rT} \tilde{E}(K - \tilde{S}($~~

0

$$-Ce^{rT} + (K - S(T))^{+}$$

$R(S(T)) = (K - S(T))$ payoff

$(K - S(T))^+ - Ce^{rT}$ gain

(S(T): R.V.)

$C = e^{-rT} \tilde{E}(K - \tilde{S}(T))^+$ Th 5.3

(C ↑ constant)

E.g3 Slide 36

$S(t) = S e^{\mu t + \sigma W_t}$ GBM

$R(T) = \frac{1}{T}\int_0^T S(t)\,dt$ ← the payoff function all values of $S(t)$, $t \in (0, T)$

(R(T) ↑ payoff)

A: By Th 5.2, $C = e^{-rT}\tilde{E}(R(T)) = e^{-rT}\tilde{E}\left(\frac{1}{T}\int_0^T S(t)\,dt\right)$ (14)

~~By Th 5.3~~,

$= \frac{e^{-rT}}{T}\tilde{E}\left(\int_0^T S(t)\,dt\right)$

By Th 5.3 $\tilde{E}\left(\int_0^T s(t)\,dt\right) = E\left(\int_0^T \tilde{S}(t)\,dt\right)$

Fact: $E\left(\int_0^T \tilde{S}(t)\,dt\right) = \int_0^T E(\tilde{S}(t))\,dt$

$E(\tilde{S}(t)) = E\left(S e^{\tilde{\mu}t + \sigma W_t}\right) \overset{*}{=} S e^{\tilde{\mu}t + \frac{\sigma^2}{2}t}$

(*) Week 1 Slide 29 Th 1.1

Since $\tilde{\mu} + \frac{\sigma^2}{2} = r$ (Th 5.1)

$E(\tilde{S}(t)) = S e^{rt}$

$\int_0^T E(\tilde{S}(t))\,dt = \int_0^T S e^{rt}\,dt = \frac{S}{r}(e^{rT} - 1)$

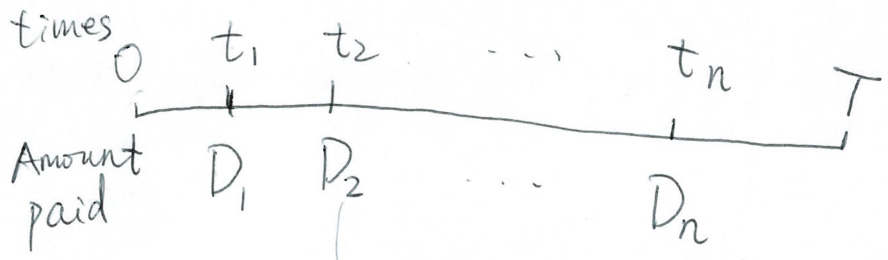
$C = \frac{e^{-rT}}{T} \times \frac{S}{r}(e^{rT} - 1) = \frac{S}{rT}(1 - e^{-rT})$

Def of dividend Slide 27

8.1 → Q1 Different types of dividends

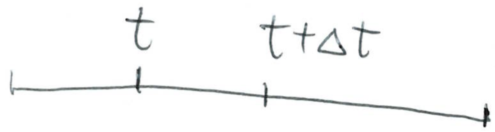
Types of dividends

① Discrete absolute dividends



② Discrete proportional dividends → i $\quad d_1 S(t_1) \quad d_2 S(t_2) \quad d_n S(t_n)$

③ Continuous proportional dividends → r



If $\Delta t > 0$ is a small time interval

then the amount paid from t to $t+\Delta t$ is $q S(t+\Delta t)\Delta t$

Assumptions:

① Dividends can be paid either in cash or shares

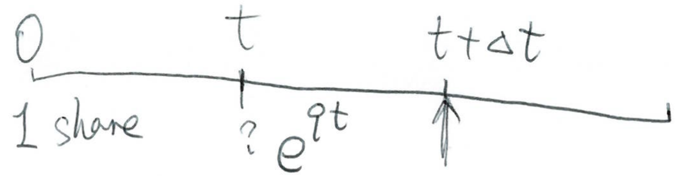
~~Assump~~

② A dividend will be re-invested in the underlying share

8.2 Continuous dividend rates Type 3

Lemma 8.1: how many shares do we own?

Suppose that the dividend is paid continously and is reinvested in the share.



dividend: $q \cdot S(t+\Delta t) \cdot \Delta t$ $\Delta t \to 0$

Stock price : $S(t+\Delta t)$

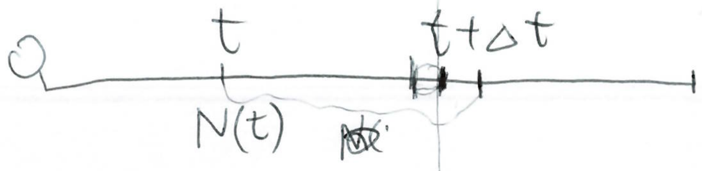
$N(t)$: the number of sharers at time t

$N(0) = 1$

$N(t) = e^{qt}$

r e^{rt}

Def
Dividend: $q\,N(t)\,S(t+\Delta t)\,\Delta t$



number to be reinvested: $\dfrac{\text{Dividend}}{\text{price}}$

$$= \frac{q\,N(t)\,S(t+\Delta t)\cdot\Delta t}{S(t+\Delta t)}$$

$$= q\,N(t)\,\Delta t$$

before paying dividend	after paying dividend
$N(t+\Delta t)$	$N(t+\Delta t) + qN(t)\Delta t$
$\approx N(t)$	$\approx N(t) + qN(t)\Delta t$

reinvestment

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = qN(t) \Rightarrow N'(t) = qN(t)$$

Number of shares at $t+\Delta t$:

$$N(t+\Delta t) = N(t) + qN(t)\Delta t$$

↑
Def

W5 ③