Week 4 Risk models with Reinsurance

Proportional reinsurance

N: num of claims — the same for reinsurer & insurer

X_i: claim amount — insurer: αX_i

reinsurer: $(1-\alpha) X_i$

S: aggregate claim amount — insurer: αS

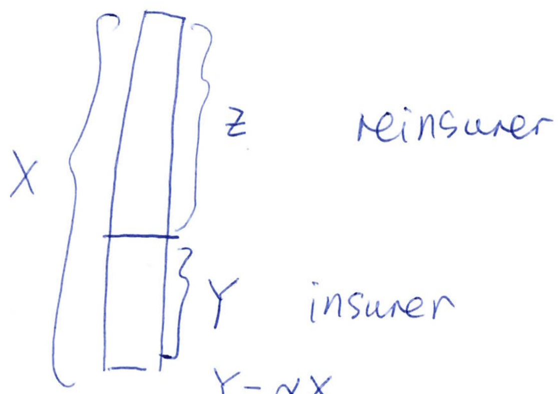
reinsurer: $(1-\alpha) S$

Example: slide 5

k prove: $M_Y(t) = M_X(kt)$

Find $M_{S_{net}}(t)$

$Y = kX$



$Y = \alpha X$

$Z = (1-\alpha) X$

$$M_Y(t) \stackrel{def}{=} E(e^{tY}) = E(e^{tkX}) = E(e^{(tk)X}) = M_X(kt)$$

S_{net} : aggregate claim amount for insurer

$$S_{net} = Y_1 + \cdots + Y_N = \sum_{i=1}^{N} Y_i$$

$$M_N(t) = e^{\lambda(e^t - 1)}$$

$$M_{S_{net}}(t) = M_N(\ln M_Y(t)) = e^{\lambda(e^{\ln M_Y(t)} - 1)}$$

$$= e^{\lambda(M_Y(t) - 1)} = e^{\lambda(M_X(kt) - 1)}$$

XOL Reinsurance

insurer:

$$Y_i = \begin{cases} X_i & X_i < M \\ M & X_i \geq M \end{cases}$$

Reinsurer:

$$Z_i = \begin{cases} 0 & X_i < M \\ X_i - M & X_i \geq M \end{cases}$$

$$S_I = Y_1 + Y_2 + \cdots + Y_N = \sum_{i=1}^{N} Y_i \qquad S_R = Z_1 + Z_2 + \cdots + Z_N = \sum_{i=1}^{N} Z_i$$

$$S = S_I + S_R$$

A. Slide 9

(a) $E(S_I)$ $E(Y_i)$

$$Y_i = \begin{cases} X_i & X_i < M \\ M & X_i \geq M \end{cases}$$

$X \sim U(a,b)$

$f_X(x) = \frac{1}{b-a}$

$$E(Y_i) = \int_0^M x f(x)\,dx + \underline{\int_M^{2000} M f(x)\,dx}$$

$X \sim U(0, 2000)$

$$= \int_0^{1600} x \cdot \frac{1}{2000}\,dx + 1600\, P(X_i > 1600)$$

$f(x) = \frac{1}{2000-0}$

$= 0.0005$

$$= \frac{1}{2000} \underbrace{\int_0^{1600} x\,dx} + 1600\, \underbrace{P(X_i > 1600)}$$

0 1600 2000

M

$$= \frac{1}{2000}\left[\frac{1}{2}x^2\right]_0^{1600} + 1600 \times \frac{2000-1600}{2000-0}$$

$$= \underline{960}$$

$$E(S_I) = \lambda\, E(Y_i)$$

$$= 10 \times 960$$

$$= 9600$$

Slide 11

W3

$E(S_I) = \underbrace{E(N)} \cdot E(Y_i)$

$= \lambda \cdot E(Y_i)$

$N \sim Poi$

$E(S_I) = \lambda\, E(Y_i)$

$= \underline{\lambda m_1}$

$Var(S_I) \leftarrow \cancel{Var(Y_i)} \leftarrow E(Y_i^2)$

$$E(Y_i^2) = \int_0^M x^2 f(x)\,dx + \int_M^{2000} M^2 f(x)\,dx$$

$$= \int_0^M x^2 \frac{1}{2000}\,dx + M^2 P(X_i > M)$$

$$= 1\,194\,666.7$$

$$Var(S_I) = \lambda E(Y_i^2) = 10 \times 1194666.7 = 11946667$$

$Skew(S_I) \leftarrow E(Y_i^3)$

$$E(Y_i^3) = \int_0^M x^3 f(x)\,dx + \int_M^{2000} M^3 f(x)\,dx$$

$$= 1\,638\,400\,000$$

$$skew(S_I) = \lambda E(Y_i^3) = 10 \times 1638400000 = \ldots$$

(b) S_R

Method 1:

$$E(z_i) = \int_0^M 0\, f(x)\, dx + \int_M^{2000} (X-M)\, f(x)\, dx$$

$$\boxed{z_i = \begin{cases} 0 & X_i < M \\ X_i - M & X_i \geq M \end{cases}}$$

Method 2:

$$\frac{a+b}{2}$$

$$E(S_I) + E(S_R) = E(S) \qquad S = S_I + S_R \qquad X_i \sim U(0, 2000)$$

$$E(X_i) = 1000$$

$$E(S_R) = E(S) - E(S_I) = \lambda E(X_i) - 9600$$

$$= 10 \times 1000 - 9600 = 400$$

$$Var(S_R) \leftarrow E(z_i^2)$$

$$E(z_i^2) = \int_0^M 0^2 f(x)\, dx + \int_M^{2000} (x-M)^2 f(x)\, dx$$

$$= \frac{1}{2000}\int_0^{2000-M} t^2\, dt = 10,666.7 \qquad t = x - M$$

$$Var(S_R) = \lambda E(z_i^2) = 10 \times 10,666.7 = 106667$$

$Skew(S_R) \leftarrow E(Z_i^3)$

$$E(Z_i^3) = \int_0^M 0^3 f(x)\,dx + \int_M^{2000} (x-M)^3 f(x)\,dx$$

$$= 3{,}200{,}000$$

$t = x - M$

$$Skew(S_R) = \lambda E(Z_i^3) = 32000000$$

$$\text{Coeff of Skew} = \frac{Skew(S_R)}{[Var(S_R)]^{\frac{3}{2}}} = 0.92$$

(c) $Var(S) \leftarrow E(X_i^2)$ without Reins

$$E(X_i^2) = \int_0^{2000} x^2 \cdot f(x)\,dx = \frac{4000{,}000}{3}$$

$$Var(S) = \lambda E(X_i^2) = 10 \times \frac{4{,}000{,}000}{3} = 13{,}333{,}333$$

(d) $Var(S_I) + Var(S_R) \neq Var(S)$ ✓

$$Var(S) = Var(S_I + S_R) = Var(S_I) + Var(S_R) + \underbrace{2Cov(S_I, S_R)}_{\neq 0}$$

S_I and S_R are <u>not independent</u>

A Slide 22

(a) without reinsurance

$X \sim \text{Pareto}(\alpha, \lambda)$ $\quad \alpha = 3,\ \lambda = 1000$

$$E(X) = \frac{\lambda}{\alpha - 1} = 500, \quad Var(X) = \frac{\alpha \lambda^2}{(\alpha-1)^2(\alpha-2)} = 750,000$$

$$E(X^2) = Var(X) + (E(X))^2 = 750,000 + 500^2$$

Poisson parameter is μ

$$E(S) = \mu E(X) = 500\mu$$

$$Var(S) = \mu E(X^2) = 1,000,000\,\mu$$

Loading factor 0.2

Total premium $= (1+0.2)\cdot \cancel{E(S)}\ E(S) = 1.2 \times 500\mu = 600\mu$

Expected profit $= \cancel{600\mu}$ Total premium $- E(S)$

$= 600\mu - 500\mu = 100\mu$

(b) With ~~Re~~ Reinsurance

$$Z_i = \begin{cases} 0 & X_i < 1000 \\ X_i - 1000 & X_i \geq 1000 \end{cases}$$

$E(S_R) \leftarrow E(Z_i)$

$$E(Z_i) = \int_0^{1000} 0 \cdot f(x)\,dx + \int_{1000}^{\infty} (x-1000)\cdot f(x)\,dx$$

$$= \int_{1000}^{\infty} (x-1000)\,\frac{3 \times 1000^3}{(1000+x)^4}\,dx$$

$u = x - 1000$

$$= \int_0^\infty u \frac{3 \times 1000^3}{(2000+u)^4} \, du$$

$$= \frac{1000^3}{2000^3} \underbrace{\int_0^\infty u \boxed{\frac{3 \times 2000^3}{(2000+u)^4}} \, du}_{\text{mean of Pareto} = \frac{\lambda}{\alpha - 1}}$$

$\boxed{\frac{3 \times 2000^3}{(2000+u)^4}} \rightarrow$ PDF Pareto $(3, 2000)$

$$\text{PDF}_{\text{Pareto}} = \frac{\alpha \lambda^\alpha}{(\lambda + x)^{\alpha+1}}$$

$$= \left(\frac{1}{2}\right)^3 \times \frac{2000}{3-1} = 125$$

$E(S_R) = \mu E(z_i) = 125\mu$

reinsurance premium $= 1.3 E(S_R) = 1.3 \times 125\mu = 162.5\mu$

Expected profit with reinsurance $= E[$ Gross premium $-$ reinsurance premium $-$ net claim amount $]$

$$= \underset{(a)}{600\mu} - 162.5\mu - \underbrace{E(S_I)}_{375\mu} = 62.5\mu$$

$E(S_I) = E(S - S_R) = E(S) - E(S_R) = \underset{(a)}{500\mu} - 125\mu = 375\mu$

W4 9

$E[\text{Profit without reinsurance}]$ (a) $= 100\mu$

$E[\text{Profit with reinsurance}]$ (b) $= 62.5\mu$ $\quad \downarrow 37.5\%$

Q: why $\downarrow 37.5\%$ in profit ?

why reinsurance ?

· 0.3 loading factor

· Cede part of the Business

$\downarrow$ risk $\leftarrow Var(S_I) < Var(S)$

Capital constraints

$\hookrightarrow$ ~~externa~~ ~~ext~~ externality

Paremeter variability

Assume we know the parameter of N $\quad N \sim Poi(\lambda)$

① Variability in a $\boxed{\text{heterogeneous}}$ portfolio

$\rightarrow$ λs are not known

λs can be different $\quad$ groups $\quad$ Motor insurance portfolio

e.g. $P(\lambda_i = 0.1) = 0.5$ $\quad$ good drivers $\quad$ better driving ability

$P(\lambda_i = 0.3) = 0.5$ $\quad$ bad drivers

② Variability in a $\boxed{\text{homogeneous}}$ portfolio

$\begin{cases} \lambda \text{ is not known} \qquad \lambda \sim \text{distribution} \\ \lambda \text{ is } \underline{\text{the same}} \text{ for all policies in the portfolio} \end{cases}$

e.g. $P(\lambda = 0.1) = 0.5$ buildings in a certain area

$P(\lambda = 0.3) = 0.5$

$N \sim Poi(\bar{\lambda}) \xrightarrow{\text{upgrade}} \begin{cases} N \sim Poi(\lambda) \\ \underset{\substack{\uparrow \\ \text{r.v.}}}{\lambda} \sim \text{distribution} \end{cases}$ nested / embedded model