Mathematical Tools for Asset Management MTH6134

Measures of Investment Risk

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Plan

- Commonly Used Measures of Investment Risks
  - Expected Value of Returns
  - Variance of Returns
  - Semi-Variance of Returns
  - Shortfall Probabilities
  - Value at Risk
  - Expected Shortfall

Measures of Investment Risk

Question: How can we rank investments/gambles/lotteries if

- if **we don't know the whole distribution of returns of investment/asset**

Answer: Use **partial known information on the distribution of returns** (i.e. moments of distribution)

Measures of Investment Risk

Return on Asset: percentage increase in the market value of an asset over a specified period

- Discrete Random Variable X can take values $x_{1,} \dots , x_n$ with probabilities $p_1 \dots , p_n$ and $\sum_i p_i = 1$
- Continuous Random Variable X can take values across a range characterised by a p.d.f. $f(x)$

Measures of Investment Risk

- Expected Value/Mean

$$E(X) \equiv \mu = \sum_i p_i x_i \text{ if } X \text{ is discrete}$$

$$E(X) \equiv \mu = \int_{-\infty}^{\infty} x f(x) dx \text{ if } X \text{ is continuous}$$

- Measures of investment risk:
  - variance of returns
  - downside semi-variance of return
  - shortfall probabilities
  - value at risk/tail value at risk

Variance of Returns

Most theories of investment risk use variance of return as the measure of risk

$$Var\left(X\right) \equiv \sigma^2 = \sum_i (x_i - \mu)^2 p_i \text{ if } X \text{ is discrete}$$

$$Var\left(X\right) \equiv \sigma^2 = \int_{-\infty}^{\infty} \left(x - \mu\right)^2 f\left(x\right) dx \text{ if } X \text{ is continuous}$$

Variance of Returns

The variance of returns $\mathrm{Var}(R) = \mathbb{E}\left((R - \mathbb{E}(R))^2\right)$

- measures uncertainty in terms of scatter around the expectation,
- measures distance between realised and expected return $R - \mathbb{E}(R)$,
- by the square the sign vanishes and larger deviations are weighted higher than smaller ones,
- by taking the outer expectation, the deviations are weighted according to their likelihoods.
- The variance is 0 if there is no risk!

Example

The investment annual returns X for a particular stock are modelled using a pdf:

$$
\begin{aligned}
f(x) &= 750\left(0.01-(x-0.05)^2\right) \\
-0.05 &\leq x \leq 0.15 \text{ or } -5\% \leq x \leq 15\%
\end{aligned}
$$

Verify that $f(x)$ is a proper pdf:

$$
\begin{aligned}
\int_{-0.05}^{0.15} f(x)\,dx &= \int_{-0.05}^{0.15} 750\left(0.01-(x-0.05)^2\right) dx \\
&= 750\int_{-0.05}^{0.15}\left(0.0075+0.1x-x^2\right) dx \\
&= 750\left[0.0075x+\frac{0.1x^2}{2}-\frac{x^3}{3}\right]_{-0.05}^{0.15} \\
&= 1
\end{aligned}
$$

Example

The average of the returns:

$$\begin{aligned}
E\left(X\right) &= 750\int_{-0.05}^{0.15} x\left(0.01-\left(x-0.05\right)^2\right)dx \\
&= 750\int_{-0.05}^{0.15} x\left(0.01-\left(x^2-0.1x+0.0025\right)^2\right)dx \\
& 750\int_{-0.05}^{0.15} \left(0.0075x+0.1x^2-x^3\right)dx \\
&= 750\left[\frac{0.0075}{2}x^2+\frac{0.1x^3}{3}-\frac{x^4}{4}\right]_{-0.05}^{0.15} = 0.05
\end{aligned}$$

Example

The variance of the returns for the same stock:

$$
\begin{aligned}
Var(X) &= \int_{-0.05}^{0.15} 750\left(0.01 - (x-0.05)^2\right)(x-0.05)^2\,dx \\
&= 750\int_{-0.05}^{0.15}\left(0.01(x-0.05)^2 - (x-0.05)^4\right)dx \\
&= 750\left[\frac{0.01}{3}(x-0.05)^3 - \frac{1}{5}(x-0.05)^5\right]_{-0.05}^{0.15} = 0.002
\end{aligned}
$$

Variance of Returns - Further Examples

Consider two assets with today's price £1,000 and the following distribution for the price after 1 week:

	25%	25%	50%
Asset 1	£ 750	£1,000	£ 1,175
Asset 2	£ 550	£1,000	£ 1,275

- What is the expected return?
- What is the variance?
- Which asset would you prefer to reduce your risk?

$$\mathbb{E}(R_1) = -0.25 * 0.25 + 0 * 0.25 + 0.175 * 0.5 = 0.0250$$
$$\mathbb{E}(R_2) = -0.45 * 0.25 + 0 * 0.25 + 0.275 * 0.5 = 0.0250$$

$$\mathrm{Var}(R_1) = \mathbb{E}[R_1^2] - \mathbb{E}[R_1]^2 = 0.0303$$
$$\mathrm{Var}(R_2) = \mathbb{E}[R_2^2] - \mathbb{E}[R_2]^2 = 0.0878$$

The variance of Asset 1 is smaller, and the expected returns are equal. Asset 1 with the smaller variance is preferred.

Variance of Returns

Consider two assets with today's price £1,000 and the following distribution for the price after 1 week:

	25%	25%	50%
Asset 1	£ 750	£1,000	£ 1,175
Asset 2	£ 800	£1,000	£ 1,500

- What is the expected return?
- What is the variance?
- Which asset would you prefer to reduce your risk?

$$\mathbb{E}(R_1) = 0.025, \qquad \mathbb{E}(R_2) = 0.2$$
$$\mathrm{Var}(R_1) = 0.0303, \qquad \mathrm{Var}(R_2) = 0.095$$

The variance of Asset 2 is higher, but its expected return as well.

Variance of Returns

Today's price S_0: £1,000

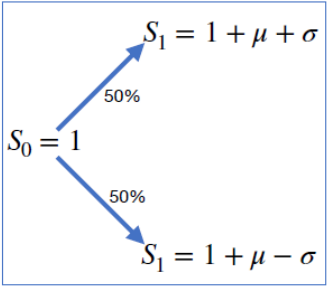
Tomorrow's price S_1, see table

	25%	25%	50%	$\mathbb{E}(R_0)$	$\text{Var}(R_0)$
Asset 1	£ 750	£1,000	£ 1,175	0.025	0.0303
Asset 2	£ 550	£1,000	£ 1,275	0.025	0.0878
Asset 3	£ 800	£1,000	£ 1,500	0.2	0.0950

For an investment decision consult at least both the returns'

- variance, and
- return!

Variance of Returns: A binomial model



Expectation and variance:

$$\mathbb{E}(S_1) = 1 + \mu$$

$$\mathrm{Var}(S_1) = \sigma^2$$

Variance of Returns: A binomial Model

Examples:

$S_0 = 1$	50%	50%
S_1	$1+\mu-\sigma$	$1+\mu+\sigma$
R_0	$\mu-\sigma$	$\mu+\sigma$

$$R_0 = S_1/S_0 - 1$$

	unlucky (50%)	lucky (50%)	$\mathbb{E}(R_0)$	$\mathrm{Var}(R_0)$
1) $\mu=0.1,\ \sigma=0$	1.1	1.1	0.1	0
2) $\mu=0.1,\ \sigma=1$	0.1	2.1	0.1	1
3) $\mu=0.07,\ \sigma=0$	1.07	1.07	0.07	0
4) $\mu=0.2,\ \sigma=0.2$	1.0	1.4	0.2	0.04
5) $\mu=0.15,\ \sigma=0.1$	1.05	1.25	0.15	0.01

Variance of Returns: A binomial Model

Pairwise comparison

$S_0 = 1$	50%	50%
R_0	$\mu - \sigma$	$\mu + \sigma$

	unlucky (50%)	lucky (50%)	$\mathbb{E}(R_0)$	$\mathrm{Var}(R_0)$
1) $\mu = 0.1, \sigma = 0$	1.1	1.1	0.1	0
2) $\mu = 0.1, \sigma = 1$	0.1	2.1	0.1	1

Which asset would you prefer?

Nr 1 as we wish to avoid unnecessary risk

Variance of Returns: A binomial Model

Pairwise comparison

$S_0 = 1$	50%	50%
R_0	$\mu - \sigma$	$\mu + \sigma$

	unlucky (50%)	lucky (50%)	$\mathbb{E}(R_0)$	$\mathrm{Var}(R_0)$
1) $\mu = 0.1, \sigma = 0$	1.1	1.1	0.1	0
3) $\mu = 0.07, \sigma = 0$	1.07	1.07	0.07	0

Which asset would you prefer?

Nr 1 as it yields more profit with the same risk

Variance of Returns: A binomial Model

Pairwise comparison:

$S_0 = 1$	50%	50%
R_0	$\mu - \sigma$	$\mu + \sigma$

	unlucky (50%)	lucky (50%)	$\mathbb{E}(R_0)$	$\mathrm{Var}(R_0)$
1) $\mu = 0.1, \sigma = 0$	1.1	1.1	0.1	0
4) $\mu = 0.2, \sigma = 0.2$	1.0	1.4	0.2	0.04

Which asset would you prefer?

Undecided. 4) has more risk, but rewards by a larger expectation

Variance of Returns: A binomial Model

Pairwise comparison:

$S_0 = 1$	50%	50%
R_0	$\mu - \sigma$	$\mu + \sigma$

	unlucky (50%)	lucky (50%)	$\mathbb{E}(R_0)$	$\text{Var}(R_0)$
2) $\mu = 0.1, \sigma = 1$	0.1	2.1	0.1	1
4) $\mu = 0.2, \sigma = 0.2$	1.0	1.4	0.2	0.04

Which asset would you prefer?

4) has a larger expectation and less risk

Variance of Returns: A binomial Model

Pairwise comparison:

$S_0 = 1$	50%	50%
R_0	$\mu - \sigma$	$\mu + \sigma$

	unlucky (50%)	lucky (50%)	$\mathbb{E}(R_0)$	$\text{Var}(R_0)$
4) $\mu = 0.2, \sigma = 0.2$	1.0	1.4	0.2	0.04
5) $\mu = 0.15, \sigma = 0.1$	1.05	1.25	0.1	0.01

Which asset would you prefer?

Undecided. 5) has less risk, but also a lower expectation

Variance of Returns: A binomial Model

	unlucky (50%)	lucky (50%)	$\mathbb{E}(R_0)$	$\text{Var}(R_0)$
1) $\mu = 0.1, \sigma = 0$	1.1	1.1	0.1	0
2) $\mu = 0.1, \sigma = 1$	0.1	2.1	0.1	1
3) $\mu = 0.07, \sigma = 0$	1.07	1.07	0.07	0
4) $\mu = 0.2, \sigma = 0.2$	1.0	1.4	0.2	0.04
5) $\mu = 0.15, \sigma = 0.1$	1.05	1.25	0.15	0.01

(1) dominates (2) and (3) and is dominated by no other
(2) dominates no other, but is dominated by (1), (4) and (5)
(3) dominates no other and is dominated by (1)
(4) dominates (2) and is dominated by no other
(5) dominates (2) and is dominated by no other

If you had to choose one investment, only (1), (4) or (5) are reasonable

Investment Dominance

Investment 1 (mean μ_1, SD σ_1) **strictly dominates** investment 2 (mean μ_2, SD σ_2), if

$$\mu_1 \geq \mu_2$$
$$\sigma_1 \leq \sigma_2$$

with one of the inequalities being strict (i.e. $\neq$).

We write $(\mu_1, \sigma_1) \succ (\mu_2, \sigma_2)$

At least one of these conditions:

(a)	(b)
$\mu_1 > \mu_2$	$\mu_1 \geq \mu_2$
$\sigma_1 \leq \sigma_2$	$\sigma_1 < \sigma_2$

Note: this is NOT Stochastic dominance!

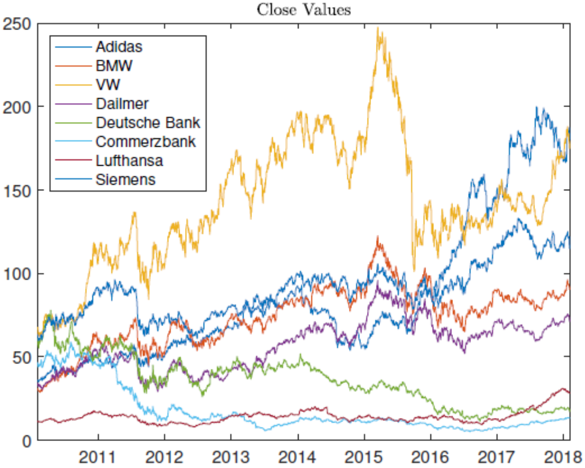
Efficient Subset

Investments that are not dominated form the *efficient subset:*

The **efficient subset** A_{eff} of $A = \{(\mu_i, \sigma_i), i \in \mathcal{I}\}$ consists of all elements $(\mu_i, \sigma_i), i \in \mathcal{I}$, which are not dominated by any other element of A.

I.e. $(\hat{\mu}, \hat{\sigma}) \in A_{\text{eff}}$ iff $(\hat{\mu}, \hat{\sigma}) \in A$ and for no $i \in \mathcal{I}$ it holds $(\mu_i, \sigma_i) \succ (\hat{\mu}, \hat{\sigma})$.

Example data

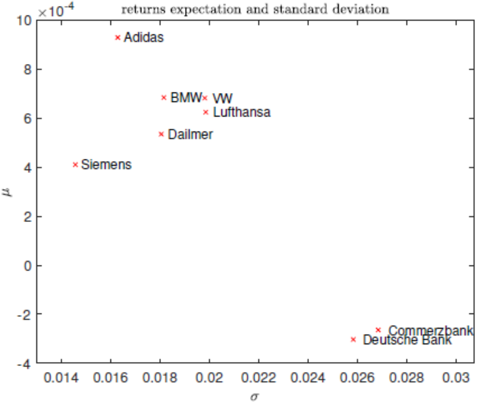


Looking at Data

If we can estimate the means and the standard deviations of returns from the data and plot them on a diagram:

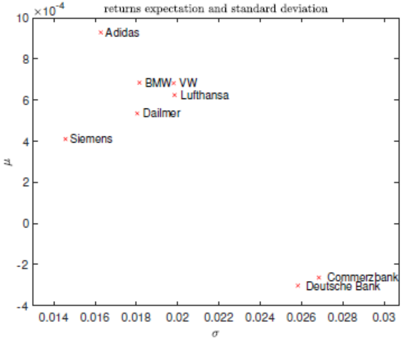
- x axis: standard deviation
- y axis: mean or expected return

Looking at Data



Warning: these values are based on historic prices. There is no guarantee that these are also the future mean and variance!

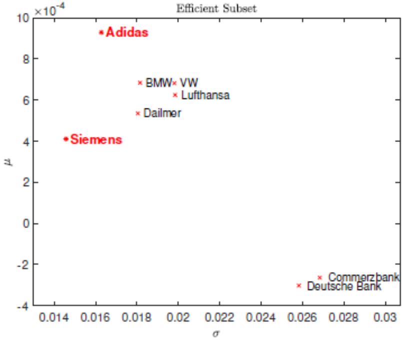
Looking at Data



Sample Exam Question:

As an investor, you want to minimise risk and maximise your profit. Based on the given data, which stocks would you invest in, i.e. determine the efficient subset?

Looking at Data



Eliminating all stocks that are dominated,

Adidas and Siemens

remain as the efficient subset.

Variance of Returns

Some drawbacks of the variance $\mathrm{Var}(R) = \mathbb{E}\left((R - \mathbb{E}(R))^2\right)$:

- unexpected large profit contributes same as a loss

  Remember:

	25%	25%	50%	mean	variance
Asset 1	£ 750	£1,000	£ 1,175	£ 1,025	30,312.5
Asset 2	£ 800	£1,000	£ 1,500	£ 1,200	95,000

- we cannot distinguish between frequent small losses and a rare huge loss
  - how likely are large losses?
  - how large are likely losses?
- Variance follows historical prices
  - what about events not present in historic prices (e.g. huge bank crash / political changes)

Semi-Variance of Return

The *(downside) semi-variance* of return (SV) is defined as:-

$$SV(X) = \sum_{x_i \leq \mu} (x_i - \mu)^2 p_i \text{ if } X \text{ is discrete}$$

$$SV(X) = \int_{-\infty}^{\mu} (x - \mu)^2 f(x) dx \text{ if } X \text{ is continuous}$$

- It doesn't take into account the variability above the mean ('upside risk')
- It is not so easy to handle mathematically
- How does this relate to variance?

Example

Continuing the first example:

$$
\begin{aligned}
SV(X) &= \int_{-0.05}^{0.05} 750\left(0.01-(x-0.05)^2\right)(x-0.05)^2\,dx \\
&= 750\int_{-0.05}^{0.05}\left(0.01(x-0.05)^2-(x-0.05)^4\right)dx \\
&= 750\left[\frac{0.01}{3}(x-0.05)^3-\frac{1}{5}(x-0.05)^5\right]_{-0.05}^{0.05} = 0.001
\end{aligned}
$$

Anything strikes you? Why is this the case?

Shortfall Probabilities

- - A shortfall probability measures the probability of returns falling below a certain level - the risk of ruin:

$$\sum_{x<L} p_i \text{ if } X \text{ is discrete}$$

$$\int_{-\infty}^{L} f(x)dx \text{ if } X \text{ is continuous}$$

- L: the chosen benchmark level
  - an absolute level required to meet a payment
  - return on a benchmark fund
- denoted $SP(X)$ or $SF(\text{bechmark})$

Example

Continuing the example, find the shortfall probability for the stock given that the benchmark return is 0

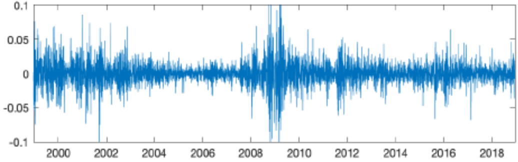
$$\begin{aligned} SP(X) &= \Pr(X < 0) = \\ &= \int_{-0.05}^{0} 750\left(0.01 - (x - 0.05)^2\right) dx \\ &= 750\left[0.01x - \frac{(x - 0.05)^3}{3}\right]_{-0.05}^{0} = 0.15625 \end{aligned}$$

Shortfall Probabilities

Shortfall probability for empirical data

Remember:

Market data:



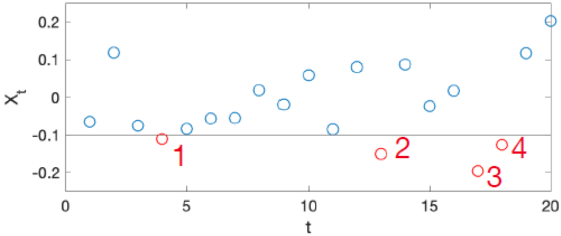
If the benchmark is $b = -0.1$:

Count:

Number of large losses vs number of all trading days

$$\mathrm{SF}_e(b) = \frac{\left|\{t : 1 \le t \le N, s.t. -X_t > b\}\right|}{N}$$

Example:



20 samples

4 samples with X <-0.1

$\Rightarrow \mathrm{SF}_e(0.1) = 20\,\%$

Value at Risk

Value at Risk (VaR):

- statistical measure of the downside risk
- uses confidence limits to assess the potential losses on a portfolio over a given future time period
- the largest number L such that the probability that the loss on the portfolio is greater than VaR, is q
- relates to Shortfall Probability but specifies a probability q and calculates the corresponding shortfall

If X is discrete:

$$VaR\left(X;q\right) = -L \text{ where } L = \left\{\max x_i : \Pr\left(X < x_i\right) \leq q\right\}$$

If X is continuous:

$$VaR\left(X;q\right) = -L \text{ where } \Pr\left(X < L\right) = q$$

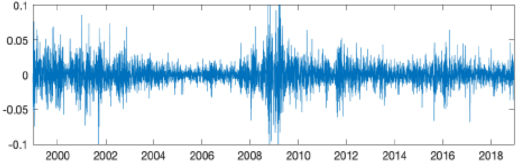
Value at Risk

- *VaR* is the mirror image of *SP*
- rather than specify a threshold value L and measure the probability, *VaR* specifies the probability and measures the corresponding threshold value
- *VaR* can be calculated from the probability of gains/losses during a period T
- *VaR* says: **We are** $100 - q$ **certain that we will not loose more than £L in time** T
- Since the nineties *VaR* a very popular measure of risk
  - JPMorgan credited with starting popularising it
  - Alternative notation $Var_{\alpha}(X)$ where α is $1 - q$

Value at Risk for empirical data

Again:

Market data:

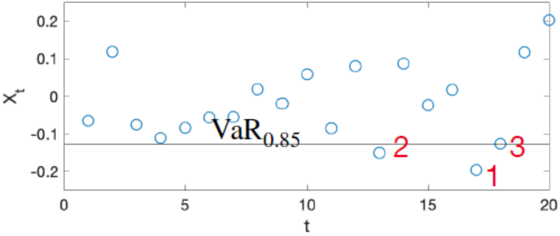


Value at Risk for empirical data

1) Sort the values by magnitude

2) Consider the smallest $(1-\alpha)N_{\text{samples}}$ elements and choose the value of the largest one:

Example: $\text{VaR}_{0.85}$



20 samples;
sort by magnitude
and count the smallest
$0.15 \cdot 20 = 3$ entries.

$\text{VaR}_{0.85} \approx 0.127$

Example

Find the *VaR* over one year with 95% confidence interval for a portfolio consisting of £100 million invested in the stock used before.

$$\Pr\left(X < L\right) = 0.05$$

$$750\int_{-0.05}^{L}\left(0.01 - \left(x - 0.05\right)^2\right)dx = 0.05$$

$$750\left(0.01x - \frac{1}{3}\left(x - 0.05\right)^3\right)\Bigg|_{-0.05}^{L} = 0.05$$

$$L = -0.02293$$

Example

Since L is a percentage investment return, the 95% value at risk on a £100 million portfolio is £100 *million* $\times$ 0.02293 =£2.293 *million*.

Interpretation: we are 95% certain that we will not loose more than £2.293 *million*.

Expected Shortfall

- *VaR* asks the question:
  - How bad things can go?
- Suppose a bank tells a trader that the one day 99% *VaR* of the trader's portfolio must be kept at less than £10 million.
  - he constructs a portfolio where 99.1% chance the daily loss is £10 million and 0.9% chance is £500 million: *Unacceptable risk*
- Expected shortfall asks the question:
  - If things go bad, what is the expected loss?

Expected Shortfall

For a shortfall probability q and corresponding threshold L such that $\Pr(X < L) = q$ then expected shortfall is:

$$E\left[\max\left(L - X, 0\right)\right] = \sum_{x_i \leq L} \left(L - x_i\right) p_i \text{ for } X \text{ discrete}$$

$$E\left[\max\left(L - X, 0\right)\right] = \int_{-\infty}^{L} \left(L - x\right) f\left(x\right) dx \text{ for } X \text{ continuous}$$

For the $(1 - q) \times 100\%$ confidence limit, expected shortfall represents the expected loss in excess of the $q - th$ lower tail value.

Example

Find the expected shortfall over one year with 95% confidence interval for a portfolio consisting of £100 million invested in the stock from the initial example.

$$
\begin{aligned}
& E\left(-0.02293 - X | X < -0.02293\right) \\
= \;& 750 \int_{-0.05}^{-0.02293} \left(-0.02293 - x\right)\left(0.01 - (x - 0.05)^2\right) dx \\
= \;& 0.000462
\end{aligned}
$$

On a portfolio of £100 million the 95% $EXSP =$£100 $million \times 0.000462 =$£0.0462 $million$.

Interpretation: the expected loss in excess of £2.293 *million* is £46, 200.