Week 4 Monday lecture 11.00-12.00

PLAN: Relativistic dynamics

Last week we introduced the concept of proper time for the relativistic motion of a particle. Let us parametrise the worldline $x^a(\tau)$ by the proper time τ then we have the following definitions

- The 4-velocity u^a is $u^a = \dfrac{dx^a}{d\tau}$
- The 4-momentum is $p^a = m u^a$
- The 4-acceleration is $a^a = \dfrac{d^2x^a}{d\tau^2} = \dfrac{du^a}{d\tau}$

Since the definition of proper time is Lorentz invariant (all inertial observers agree on $d\tau$) then u^a, p^a, a^a have the same transformation properties a x^a, i.e. they are spacetime vectors. Since $(c\,d\tau)^2 = ds^2 = (dx^0)^2 - \sum_i (dx^i)^2$ (see Week 3), we have $u^2 = u^a \eta_{ab} u^b = -c^2$. The relation with the 3-velocity is

$$u^2 = -\left(\frac{dx^0}{d\tau}\right)^2 + \frac{dx^i}{d\tau}\frac{dx^i}{d\tau} = -\left(\frac{dt}{d\tau}\right)^2\left[\left(\frac{dx^0}{dt^2}\right) - \left(\frac{dx^i}{dt}\right)\left(\frac{dx^i}{dt}\right)\right]$$

$$= -\left(\frac{dt}{d\tau}\right)^2\left[c^2 - \underline{v}^2\right] = -c^2 \Rightarrow \left(\frac{dt}{d\tau}\right) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

Thus $u^a = \frac{dt}{d\tau}\frac{dx^a}{dt} = c\gamma(1, \underline{v})$

Then the 4-momentum is $p^a = cm\gamma(1, \underline{v})$. The spatial component looks like the Newtonian formula with a velocity dependent mass

$$m(v) = \gamma m = \frac{m}{\sqrt{1-v^2/c^2}}$$

Of course the combination $p^2 = p^a \eta_{ab} p^b$ is Lorentz invariant. Thus we can calculate it in a frame where the particle is at rest and the result is universal for all inertial frames $p^2 = -(mc)^2$. The zero component of the 4-momentum is directly related to the energy of the particle : $p^0 = E/c$. Thus exactly as ct and $\underline{x}$ combine to form a 4-vector, E/c and $\underline{p}$ combine to form anothe

4-vector! We can check the this identification between E and p^0 is not crazy by looking at the small velocity limit

terms that vanish as $v/c \to 0$

$$cp^0 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = mc^2 + \frac{1}{2} m v^2 + \dots$$

rest mass energy   kinetic energy

According to classical physics only waves can propagate at the speed of light, but in quantum mechanics these waves are made out of particles! Thus we should be able to define the notion of a particle that moves at the speed of light (these particles of light are called photons), but this is possible only if they are massless (rest mass $m=0$).

In this case instead of having m and γ as well defined quantities separately only $(\gamma m c^2)$ makes sense a single object:

frequency of the wave

$$cp^0 = (\gamma m c^2) \xrightarrow[v=c]{\text{massless}} cp^0 = h\nu$$

Planck constant

Notice that the 4-momenta of a massless particle is null since it travels at $v=c$, ie $p^2=0$.

At this stage we can generalise the three laws of particle dynamics from week 1 to the case of special relativity

<u>1st law:</u> unchanged (the only difference is that there is a maximum velocity c on which all inertial observers agree)

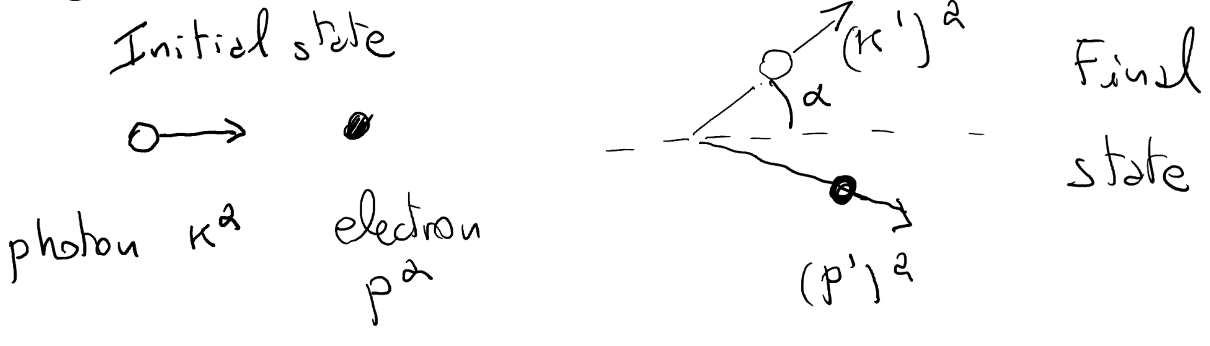
<u>2nd law</u> $$F^a = m a^a = m \frac{d^2 x^a}{d\tau^2}$$

(notice that $F^a \eta_{ab} u^b = m \frac{du^a}{d\tau} \eta_{ab} u^b = \frac{m}{2}\frac{du^2}{d\tau} = 0$

since $u^2 = -c^2$)

<u>3rd law</u> $$\sum_i p^a_{(i)} = \text{const.}$$ sum of the 4-momenta of all particles in an isolated system is constant

As an example let us consider the following case (Compton scattering): a photon scatters against an electron at rest



By applying the 3^{rd} law we have

$$k^a + p^a = (k')^a + (p')^a \Rightarrow -(k')^a + k^a + p^a = (p')^a$$

We can calculate the relativistic norm of both sides

$$\left[-(k')^a + p^a + k^a\right] \eta_{ab} \left[-(k')^b + p^b + k^b\right] = (p')^a \eta_{ab} (p')^b$$

$$\cancel{(k')^2}^{\to 0} + \cancel{k^2}^{\to 0} + \cancel{(p)^2} - 2k'p - 2kk' + 2kp = \cancel{(-p')^2}$$

since k'^a, k^a are null vectors and $p^2 = p'^2 = m_e^2 c^2$.

By using $k^a = \left(\frac{E_k}{c}, \underline{k}\right)$, $k'^a = \left(\frac{E_{k'}}{c}, \underline{k'}\right)$ and

$p^a = (m_e c, 0, 0, 0)$ with $E_k = h\nu_k$ we have

$$kk' = \frac{E_k E_{k'}}{c^2}\left(-1 + \cos\alpha\right) = \frac{h\nu_k}{c}\frac{h\nu_{k'}}{c}\left(-1 + \cos\alpha\right)$$

$k'p = -m_e c \frac{E_{k'}}{c} = -m_e E_{k'}$ and similarly for kp.

$$\underbrace{\frac{h\nu_k}{c}\frac{h\nu_{k'}}{c}(1 - \cos\alpha)}_{-kk'} = m_e\left(\underbrace{h\nu_k}_{-kp} - \underbrace{h\nu_{k'}}_{k'p}\right) \Rightarrow$$

wavelength

$\lambda\nu = c$

$$\left(\frac{c}{\nu_{k'}} - \frac{c}{\nu_k}\right) = \frac{h}{m_e c}(1 - \cos\alpha) \Rightarrow \left(\lambda_{k'} - \lambda_k\right) = \frac{h}{m_e c}(1 - \cos\alpha)$$

Thus the wavelength of the scattered light is larger than the one of the incident light by a factor that depends on the scattering angle α.