MTH5113 (2023/24): Problem Sheet 2
Solutions

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(1) *(Warm-up)* Compute each of the following:

(a) Consider the vector-valued function

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}^2, \qquad \mathbf{f}(t) = (t^2, t^3 - 1).$$

(i) Compute $\mathbf{f}'(t)$ for every $t \in \mathbb{R}$.

(ii) Find the values $\mathbf{f}'(0)$, $\mathbf{f}'(1)$, and $\mathbf{f}'(-2)$.

(b) Consider the vector-valued function

$$\mathbf{g} : (0, 1) \to \mathbb{R}^3, \qquad \mathbf{g}(t) = (\ln t, \ln(1 - t), e^{3t} + t).$$

(i) What happens to $\mathbf{g}(t)$ as t approaches 0? As t approaches 1?

(ii) Compute $\mathbf{g}'(t)$ for every $t \in \mathbb{R}$.

(iii) Compute the second derivative $\mathbf{g}''(t)$ for every $t \in \mathbb{R}$.

(a) These are direct computations:

(i) To find $\mathbf{f}'(t)$, we differentiate each component of $\mathbf{f}$:

$$\mathbf{f}'(t) = \left(\frac{d}{dt}(t^2), \frac{d}{dt}(t^3 - 1) \right) = (2t, 3t^2).$$

(ii) We substitute $t = 0$, $t = 1$, and $t = -2$ into the preceding formula:

$$\mathbf{f}'(0) = (2 \cdot 0, 3 \cdot 0^2) = (0, 0),$$
$$\mathbf{f}'(1) = (2 \cdot 1, 3 \cdot 1^2) = (2, 3),$$
$$\mathbf{f}'(-2) = (2 \cdot (-2), 3 \cdot (-2)^2) = (-4, 12).$$

(b) These are also direct computations:

(i) As t approaches 0, the x-component $(\ln t)$ of $\mathbf{g}(t)$ tends to $-\infty$, while the y- and z-components have finite limits (0 and 1, respectively):

$$\lim_{t \searrow 0} \mathbf{g}(t) = (-\infty, 0, 1).$$

Also, as t approaches 1, the y-component $(\ln(1-t))$ of $\mathbf{g}(t)$ tends to $-\infty$, while the x- and z-components have finite limits (0 and $e^3 + 1$, respectively).

$$\lim_{t \nearrow 1} \mathbf{g}(t) = (0, -\infty, e^3 + 1).$$

(ii) Recalling the calculus identities

$$\frac{d}{dt}(\ln t) = \frac{1}{t}, \qquad \frac{d}{dt}(e^t) = e^t, \qquad t \in (0, 1),$$

as well as the chain rule, we obtain that

$$\begin{aligned} \mathbf{g}'(t) &= \left(\frac{d}{dt}(\ln t),\ \frac{d}{dt}[\ln(1-t)],\ \frac{d}{dt}(e^{3t} + t) \right) \\ &= \left(\frac{1}{t}, \frac{1}{1-t} \cdot \frac{d}{dt}(1-t),\ e^{3t} \cdot \frac{d}{dt}(3t) + 1 \right) \\ &= \left(\frac{1}{t}, \frac{1}{t-1}, 3e^{3t} + 1 \right). \end{aligned}$$

(iii) To compute $\mathbf{g}''(t)$, we differentiate the preceding formula yet again:

$$\begin{aligned} \mathbf{g}''(t) &= \left(\frac{d}{dt}\left(\frac{1}{t}\right), \frac{d}{dt}\left(\frac{1}{t-1}\right), \frac{d}{dt}(3e^{3t} + 1) \right) \\ &= \left(-\frac{1}{t^2}, -\frac{1}{(t-1)^2}, 9e^{3t} \right). \end{aligned}$$

(2) *(Warm-up)* Let $\mathbf{A}$ denote the vector-valued function

$$\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^2, \qquad \mathbf{A}(x, y, z) = (x(1-z), y(1-z)).$$

(a) Compute the partial derivatives $\partial_1 \mathbf{A}(x, y, z)$, $\partial_2 \mathbf{A}(x, y, z)$, and $\partial_3 \mathbf{A}(x, y, z)$ at every point $(x, y, z) \in \mathbb{R}^3$.

(b) Find $\partial_2 \mathbf{A}(1, 0, 3)$ and $\partial_3 \mathbf{A}(0, -1, -1)$.

(a) To find $\partial_1 \mathbf{A}(x, y, z)$, we take the corresponding derivative of each component of $\mathbf{A}$:

$$\partial_1 \mathbf{A}(x, y, z) = (\partial_x[x(1-z)], \partial_x[y(1-z)]) = (1-z, 0).$$

In the last step, we treated y and z as constants and differentiated with respect to x. The remaining partial derivatives of $\mathbf{A}$ are computed analogously:

$$\partial_2 \mathbf{A}(x, y, z) = (\partial_y[x(1-z), \partial_y[y(1-z)]]) = (0, 1-z),$$
$$\partial_3 \mathbf{A}(x, y, z) = (\partial_z[x(1-z), \partial_z[y(1-z)]]) = (-x, -y).$$

(Make sure your answers are 2-dimensional vectors!)

(b) Substituting $(x, y, z) = (1, 0, 3)$ to the above formula for $\partial_2 \mathbf{A}(x, y, z)$ yields

$$\partial_2 \mathbf{A}(1, 0, 3) = (0, 1-3) = (0, -2).$$

Similarly, substituting $(x, y, z) = (0, -1, -1)$ into the formula for $\partial_3 \mathbf{A}(x, y, z)$ yields

$$\partial_3 \mathbf{A}(0, -1, -1) = (-0, -(-1)) = (0, 1).$$

(3) *(Warm-up)* Let $\mathbf{F}$ be the vector field on $\mathbb{R}^2$ defined via the formula

$$\mathbf{F}(x, y) = (x-y, x+y)_{(x,y)}.$$

(a) Compute the following: (i) $\mathbf{F}(1, -1)$; (ii) $\mathbf{F}(-2, -1)$; (iii) $\mathbf{F}(-1, \frac{1}{2})$.

(b) Plot the three tangent vectors from part (a) onto a Cartesian plane.

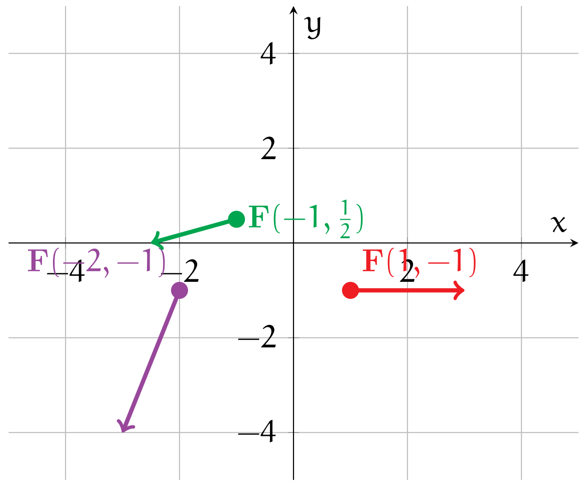
(a) Each of these is a direct computation:

(i) $\mathbf{F}(1, -1) = (1-(-1), 1+(-1))_{(1,-1)} = (2, 0)_{(1,-1)}$.

(ii) $\mathbf{F}(-2, -1) = (-2-(-1), -2+(-1))_{(-2,-1)} = (-1, -3)_{(-2,-1)}$.

(iii) $\mathbf{F}(-1, \frac{1}{2}) = (-1-\frac{1}{2}, -1+\frac{1}{2})_{(-1,\frac{1}{2})} = (-\frac{3}{2}, -\frac{1}{2})_{(-1,\frac{1}{2})}$.

(b) The tangent vectors are drawn below:



(4) **[Tutorial]** Consider the following vector-valued function:

$$\mathbf{h} : (0, \infty) \to \mathbb{R}^2, \qquad \mathbf{h}(t) = (t \cos t,\ t \sin t).$$

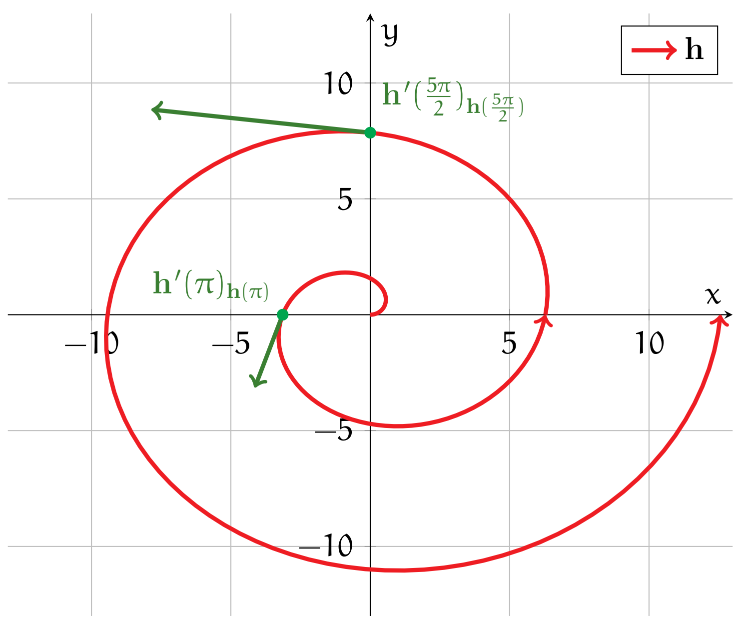
(a) Sketch the values $\mathbf{h}(t)$, for all $0 < t < 4\pi$. Also, plot the values of $\mathbf{h}$ on computer (see the *Additional Resources* section on the *QMPlus* page).

(b) Compute $\mathbf{h}(\pi)$ and $\mathbf{h}(\frac{5\pi}{2})$.

(c) Compute $\mathbf{h}'(\pi)$ and $\mathbf{h}'(\frac{5\pi}{2})$.

(d) Draw $\mathbf{h}'(\pi)_{\mathbf{h}(\pi)}$ and $\mathbf{h}'\left(\frac{5\pi}{2}\right)_{\mathbf{h}(\frac{5\pi}{2})}$ on your sketch in part (a).

(a) The sketch is below, with the image of $\mathbf{h}$ drawn in red.



(b) The desired values of $\mathbf{h}$ are below:

$$\mathbf{h}(\pi) = (\pi \cos \pi, \pi \sin \pi) = (-\pi, 0),$$
$$\mathbf{h}\left(\frac{5\pi}{2}\right) = \left(\frac{5\pi}{2}\cos\frac{5\pi}{2}, \frac{5\pi}{2}\sin\frac{5\pi}{2}\right) = \left(0, \frac{5\pi}{2}\right),$$

(c) Taking a derivative of $\mathbf{h}$ (using the product rule) yields

$$\mathbf{h}'(t) = (\cos t - t\sin t, \sin t + t\cos t).$$

In particular, setting $t = \pi$ and $t = \frac{5\pi}{2}$ yields

$$\mathbf{h}'(\pi) = (\cos \pi - \pi \sin \pi, \sin \pi + \pi \cos \pi) = (-1, -\pi),$$
$$\mathbf{h}'\left(\frac{5\pi}{2}\right) = \left(\cos\frac{5\pi}{2} - \frac{5\pi}{2}\sin\frac{5\pi}{2}, \sin\frac{5\pi}{2} + \frac{5\pi}{2}\cos\frac{5\pi}{2}\right) = \left(-\frac{5\pi}{2}, 1\right),$$

(d) The tangent vectors

$$\mathbf{h}'(\pi)_{\mathbf{h}(\pi)} = (-1, \pi)_{(-\pi,0)}, \qquad \mathbf{h}'\left(\frac{5\pi}{2}\right)_{\mathbf{h}(\frac{5\pi}{2})} = \left(-\frac{5\pi}{2}, 1\right)_{(0,\frac{5\pi}{2})},$$

are drawn as green arrows on the diagram in part (a).

(5) **[Marked]** Let β be the vector-valued function

$$\beta : (-2, 2) \times (0, 2\pi) \to \mathbb{R}^3, \qquad \beta(u, v) = \left\{\frac{3u}{2} - \frac{u\cos(v)}{\sqrt{1+u^2}}, \sin(v), \frac{3u^2}{4} + \frac{\cos(v)}{\sqrt{1+u^2}}\right\}.$$

(a) Sketch the image of β (i.e. plot all values $\beta(u, v)$, for (u, v) in the domain of β).

(b) On the sketch in part (a), indicate (i) the path obtained by holding $v = \pi/2$ and varying u, and (ii) the path obtained by holding $u = 0$ and varying v.

(c) Compute the following quantities:

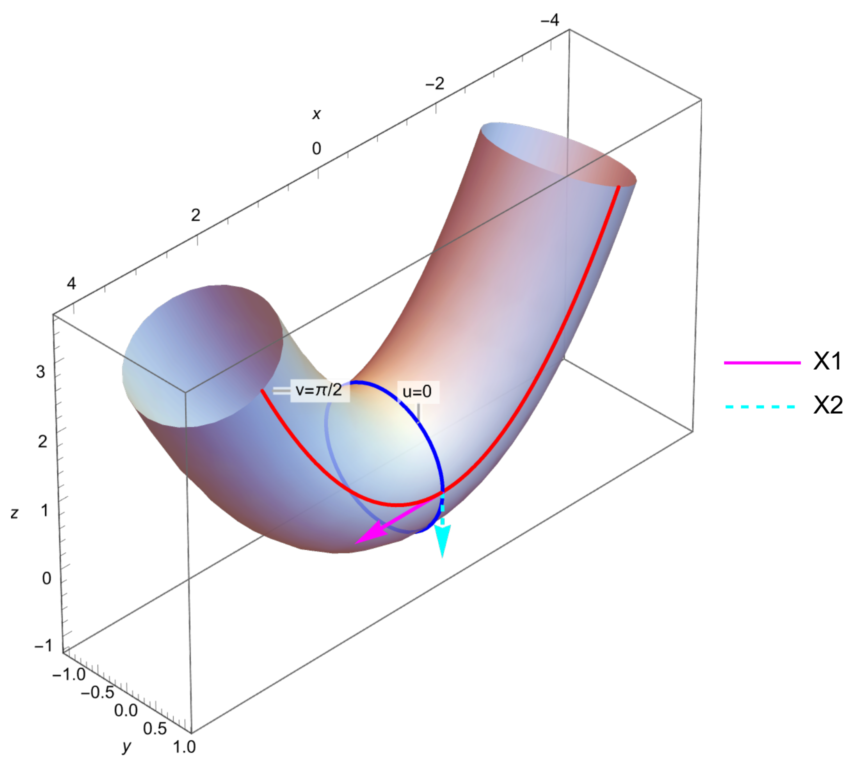
$$\beta\left(0, \frac{\pi}{2}\right), \qquad \partial_1\beta\left(0, \frac{\pi}{2}\right), \qquad \partial_2\beta\left(0, \frac{\pi}{2}\right).$$

(d) Draw the following tangent vectors on your sketch in part (a):

$$X_1 = \partial_1\beta\left(0, \frac{\pi}{2}\right)_{\beta\left(0,\frac{\pi}{2}\right)}, \qquad X_2 = \partial_2\beta\left(0, \frac{\pi}{2}\right)_{\beta\left(0,\frac{\pi}{2}\right)}.$$

(a) The image of β is drawn below. [1 mark for mostly correct drawing]

(b) The path in (i) is drawn below in red *(a parabola)*, while the path in (ii) is drawn in blue *(a circle)*. [1 mark for mostly correct drawings]



(c) We first compute the partial derivatives of β: [1 mark]

$$\partial_1\beta(u,v) = \left\{\frac{3}{2} - \frac{\cos(v)}{(1+u^2)^{3/2}}, 0, \frac{3u}{2} - \frac{u\cos(v)}{(1+u^2)^{3/2}}\right\},$$

$$\partial_2\beta(u,v) = \left\{\frac{u\sin(v)}{\sqrt{1+u^2}}, \cos(v), -\frac{\sin(v)}{\sqrt{1+u^2}}\right\}.$$

Evaluating at $(u,v) = (0, \frac{\pi}{2})$, we obtain [1 mark]

$$\beta\left(0, \frac{\pi}{2}\right) = (0,1,0), \quad \partial_1\beta\left(0, \frac{\pi}{2}\right) = \left(\frac{3}{2}, 0, 0\right), \quad \partial_2\beta\left(0, \frac{\pi}{2}\right) = (0,0,-1).$$

(d) These tangent vectors are drawn in the diagram from parts (a) and (b) (X_1 in magenta, and X_2 in dashed cyan). [1 mark for mostly correct arrows]

(6) *(Compute 'n' plot)* Let λ denote the vector-valued function

$$\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad \lambda(t) = (t, t^2 - 1).$$

(a) Compute the following: $\lambda(-2)$, $\lambda(-1)$, $\lambda(0)$, $\lambda(1)$, and $\lambda(2)$.

(b) Compute the following: $\lambda'(-2)$, $\lambda'(-1)$, $\lambda'(0)$, $\lambda'(1)$, and $\lambda'(2)$.

(c) Sketch the values $\lambda(t)$, for all $-3 < t < 3$, on a Cartesian plane.

(d) Draw the following tangent vectors as arrows on your sketch in part (a):

$$\lambda'(-2)_{\lambda(-2)}, \qquad \lambda'(-1)_{\lambda(-1)}, \qquad \lambda'(0)_{\lambda(0)}, \qquad \lambda'(1)_{\lambda(1)}, \qquad \lambda'(2)_{\lambda(2)}.$$

(a) The desired values of λ are below:

$$\begin{aligned} \lambda(-2) &= (-2, (-2)^2 - 1) = (-2, 3), \\ \lambda(-1) &= (-1, (-1)^2 - 1) = (-1, 0), \\ \lambda(0) &= (0, 0^2 - 1) = (0, -1), \\ \lambda(1) &= (1, 1^2 - 1) = (1, 0), \\ \lambda(2) &= (2, 2^2 - 1) = (2, 3). \end{aligned}$$

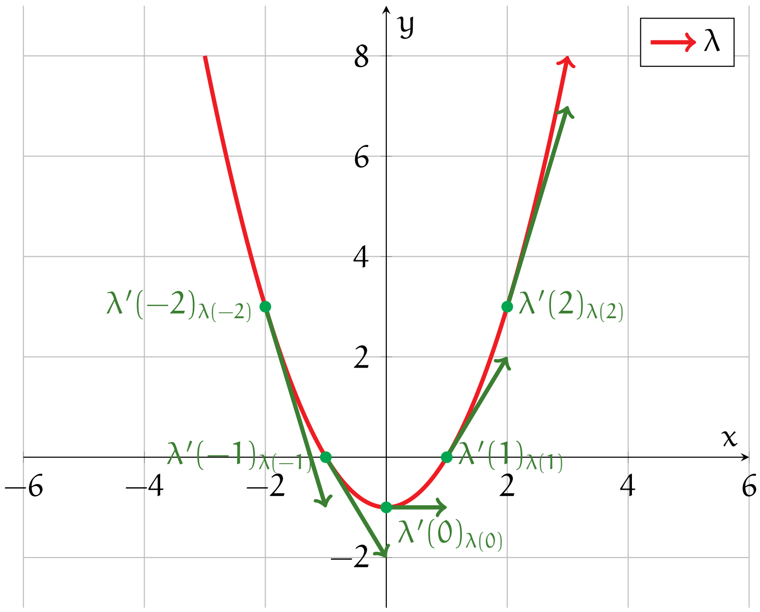
(b) First, note that the derivative of λ satisfies

$$\lambda'(t) = (1, 2t).$$

Thus, taking $t = -2, -1, 0, 1, 2$, we obtain

$$\begin{aligned} \lambda'(-2) &= (1, 2 \cdot (-2)) = (1, -4), \\ \lambda'(-1) &= (1, 2 \cdot (-1)) = (1, -2), \\ \lambda'(0) &= (1, 2 \cdot 0) = (1, 0), \\ \lambda'(1) &= (1, 2 \cdot 1) = (1, 2), \\ \lambda'(2) &= (1, 2 \cdot 2) = (1, 4). \end{aligned}$$

(c) The sketch is below, with the image of λ drawn in red. *(The most direct way to sketch this is to note that λ is the graph of the parabolic function $f(x) = x^2 - 1$. In addition, you could use the answers in parts (a) and (b) to help you draw λ.)*



(d) From parts (b) and (c), we have that

$$\lambda'(-2)_{\lambda(-2)} = (1, -4)_{(-2,3)},$$
$$\lambda'(-1)_{\lambda(-1)} = (1, -2)_{(-1,0)},$$
$$\lambda'(0)_{\lambda(0)} = (1, 0)_{(0,-1)},$$
$$\lambda'(1)_{\lambda(1)} = (1, 2)_{(1,0)},$$
$$\lambda'(2)_{\lambda(2)} = (1, 4)_{(2,3)}.$$

These are drawn as green arrows on the diagram in part (c).

(7) *(Compute 'n' plot II)* Consider the vector-valued function

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u).$$

(See also Question 8 from Problem Sheet 1.)

(a) Sketch the image of σ. (Use a computer to help if needed; see the *Additional Resources* section on the *QMPlus* page)

(b) On the sketch in part (a), indicate (i) the path obtained by holding $u = \frac{\pi}{2}$ and varying v, and (ii) the path obtained by holding $v = \frac{\pi}{2}$ and varying u.

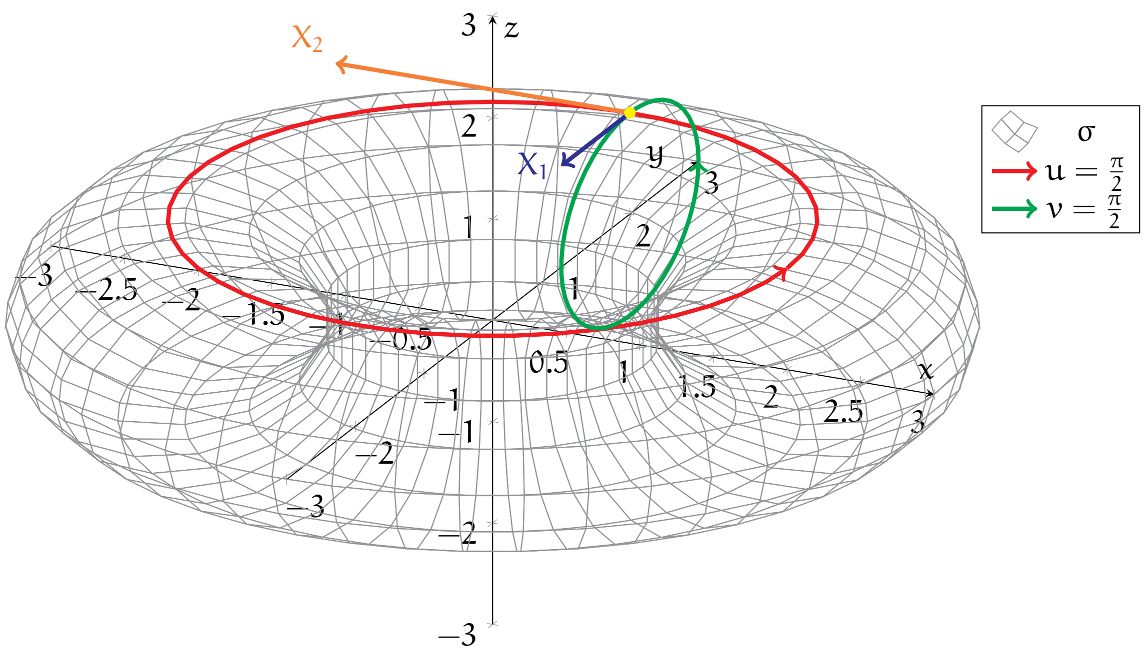
(c) Compute the partial derivatives $\partial_1\sigma(u, v)$ and $\partial_2\sigma(u, v)$ for all $(u, v) \in \mathbb{R}^2$.

(d) Draw the following tangent vectors on your sketch in part (a):

$$X_1 = \partial_1\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right)_{\sigma(\frac{\pi}{2},\frac{\pi}{2})}, \qquad X_2 = \partial_2\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right)_{\sigma(\frac{\pi}{2},\frac{\pi}{2})}.$$

(a) A sketch is given below part (b), with the image of σ drawn in grey.

(b) The path in (i) is drawn below in red, while the path in (ii) is drawn in green.



(c) To find $\partial_1\sigma(u, v)$ and $\partial_2\sigma(u, v)$, we differentiate each component:

$$\partial_1\sigma(u, v) = (-\sin u \cos v, -\sin u \sin v, \cos u),$$
$$\partial_2\sigma(u, v) = (-(2 + \cos u) \sin v, (2 + \cos u) \cos v, 0).$$

(d) First, we compute

$$\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (0, 2, 1), \qquad \partial_1\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (0, -1, 0), \qquad \partial_2\sigma\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = (-2, 0, 0).$$

As a result, we have that

$$X_1 = (0, -1, 0)_{(0,2,1)}, \qquad X_2 = (-2, 0, 0)_{(0,2,1)},$$

The tangent vectors X_1 and X_2 are drawn in the diagram from parts (a) and (b).

(8) *(Gradients 'n' plot)* Consider the function

$$p : \mathbb{R}^2 \to \mathbb{R}, \qquad p(x, y) = x - y^2.$$

(a) Sketch the following sets on a Cartesian plane:

(i) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\}$.

(ii) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 2\}$.

(iii) $\{(x, y) \in \mathbb{R}^2 \mid p(x, y) = -2\}$.

(b) Compute the gradient $\nabla p(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

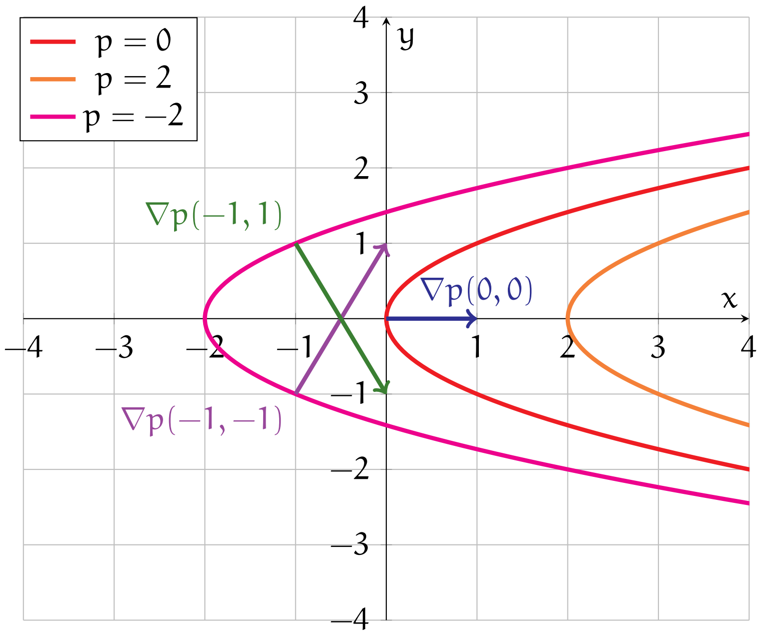
(c) Plot the following values onto your sketch from part (a):

(i) $\nabla p(0, 0)$.

(ii) $\nabla p(-1, -1)$.

(ii) $\nabla p(-1, 1)$.

(a) The three sets are sketched below in (i) red, (ii) orange, and (iii) pink:



(b) The partial derivatives of p are

$$\partial_1 p(x, y) = 1, \qquad \partial_2 p(x, y) = -2y.$$

Thus, the gradient of p is

$$\nabla p(x, y) = (\partial_1 p(x, y), \partial_2 p(x, y))_{(x,y)} = (1, -2y)_{(x,y)}.$$

(c) Substituting the appropriate values for x and y, we obtain that

(i) $\nabla p(0, 0) = (1, 0)_{(0,0)}$.

(ii) $\nabla p(-1, -1) = (1, 2)_{(-1,-1)}$.

(iii) $\nabla p(-1, 1) = (1, -2)_{(-1,1)}$.

The corresponding arrows are drawn in the plot from (a) in (i) blue, (ii) purple, (iii) green.

(9) *(Connections to "Convergence and Continuity")* Consider the following subsets of $\mathbb{R}^2$:

$$V = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}, \qquad L = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}.$$

(a) Give an informal justification of the following: (i) V is open; (ii) L is not open.

(b) *(Not examinable)* Give a rigorous proof of the two statements in part (a).

(c) Is the following subset of $\mathbb{R}^2$ connected:

$$Q = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}?$$

Give a brief (informal) justification of your answer.

(a) Informal justifications for both statements are given below:

(i) Consider a point $(x, y) \in V$, so that $x > 0$. Suppose you take a step away from (x, y), in any direction, to another point (x', y'). Then, as long as that step is small enough, we would still have $x' > 0$, and hence $(x', y') \in V$. Thus, by definition, V is open.

(ii) Consider the point $(0, 0) \in L$. Suppose you take a step away from $(0, 0)$ in the x-direction. Then, no matter how small of a step you take, you will always no longer be on L. Thus, L violates the definition of openness and hence is not open.

(b) Formal proofs of both statements are given below:

(i) To prove that V is open, we must establish the following statement:

($*$) *For any* $(x, y) \in V$*, there exists* $\delta > 0$ *such that for any* $(x', y') \in \mathbb{R}^2$ *satisfying* $|(x', y') - (x, y)| < \delta$*, we have* $(x', y') \in V$*.*

Let (x, y) be an arbitrary element of V; note that $x > 0$. Moreover, let us choose $\delta = x > 0$. Then, given any $(x', y') \in \mathbb{R}^2$ such that $|(x', y') - (x, y)| < x$, we have that

$$x > |(x', y') - (x, y)| \geq |x' - x|,$$

and it follows that $x' > 0$. As a result, $(x', y') \in V$, and hence ($*$) is proved.

(ii) Negating the definition of open subsets, we see that we must prove the following:

($\star$) *There exists some* $(x, y) \in L$ *such that for every* $\delta > 0$*, there exists* $(x', y') \in \mathbb{R}^2$ *such that* $|(x', y') - (x, y)| < \delta$*, but* $(x', y') \notin L$*.*

Let us choose $(x, y) = (0, 0) \in L$. Given an arbitrary $\delta > 0$, we choose the point $(x', y') = (\frac{\delta}{2}, 0)$. In particular, we have that $(\frac{\delta}{2}, 0) \notin L$, and that

$$|(x', y') - (x, y)| = \left| \left(\frac{\delta}{2}, 0 \right) - (0, 0) \right| = \frac{\delta}{2} < \delta.$$

In particular, the above proves the statement $(\star)$.

(c) The set Q is not connected.

To justify this, we consider two points $(x_1, y_1), (x_2, y_2) \in Q$, with $y_1 < 0 < y_2$. Then, any path that connects (x_1, y_1) to (x_2, y_2) must pass through the (horizontal) line $y = 0$ (this comes from the intermediate value theorem), and hence this path must leave Q.

(10) *(Good derivative, bad derivative)*

- **(a)** *(Not examinable)* Give an example of a function $b : \mathbb{R}^2 \to \mathbb{R}$ such that (i) $\partial_1 b(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, but (ii) $\partial_2 b(x, y)$ fails to exist for some (x, y).
- **(b)** *(Fun! But not examinable)* Give an example of a function $b : \mathbb{R}^2 \to \mathbb{R}$ such that (i) $\partial_1 b(x, y)$ exists for all $(x, y) \in \mathbb{R}^2$, but (ii) $\partial_2 b(x, y)$ fails to exist for *any* (x, y).

(a) One example of such a function b is the following:

$$b : \mathbb{R}^2 \to \mathbb{R}, \qquad b(x, y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases}$$

Note that b is always constant if we hold y constant and vary only with respect to x. As a result, $\partial_1 b(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$.

On the other hand, if we fix $x = 0$, for instance, and we vary in y, we see that

$$b(0, y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases}$$

In particular, this fails to be continuous at $y = 0$, hence we cannot differentiate with respect to y there. As a result, $\partial_2 b(0, 0)$ fails to exist.

(b) One example of such a function b is the following:

$$b : \mathbb{R}^2 \to \mathbb{R}, \qquad b(x, y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}, \\ 0 & \text{if } y \notin \mathbb{Q}. \end{cases}$$

(Here, $\mathbb{Q}$ is the set of rational numbers.)

Again, b is always constant if we hold y constant and vary only with respect to x. As a result, $\partial_1 b(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$. On the other hand, if we fix any x-value and vary in y, then the resulting function $y \mapsto b(x, y)$ fails to be continuous at any value of y. As a result, $\partial_2 b(x, y)$ cannot exist at any $(x, y) \in \mathbb{R}^2$.