MTH5113 (2023/24): Problem Sheet 4

All coursework should be submitted individually.

- Problems marked "[**Marked**]" should be submitted and will be marked.

Please submit the completed problem on QMPlus:

- At the portal for **Coursework Submission 1**.

---

(1) *(Warm-up)*

(a) Consider the following regular parametric curve:

$$\mathbf{h} : \mathbb{R} \to \mathbb{R}^3, \qquad \mathbf{h}(t) = (t, \cos t, \sin t).$$

(i) Find the tangent line to $\mathbf{h}$ at $t = \frac{3\pi}{4}$.

(ii) Sketch $\mathbf{h}$, the tangent vector $\mathbf{h}'(\frac{3\pi}{4})_{\mathbf{h}(\frac{3\pi}{4})}$, and the tangent line from (i).

(b) Consider the following regular parametric curve:

$$\mathbf{k} : (0, \infty) \to \mathbb{R}^2, \qquad \mathbf{k}(t) = (t \cos t, t \sin t).$$

(i) Find the tangent line to $\mathbf{k}$ at $t = \pi$.

(ii) Sketch $\mathbf{k}$, the tangent vector $\mathbf{k}'(\pi)_{\mathbf{k}(\pi)}$, and the tangent line from (i).

(2) *(Warm-up)* Let $\mathcal{C}$ denote the unit circle about the origin:

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Compute, at each of the points $\mathbf{p} \in \mathcal{C}$ listed below, the tangent line to $\mathcal{C}$:

(a) $\mathbf{p} = (1, 0)$.

(b) $\mathbf{p} = (0, 1)$.

(c) $\mathbf{p} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(3) *(Am I a curve?)* For each of the sets C provided below: (i) give a sketch of C, (ii) determine whether C is a curve or not, and (iii) justify your answer.

(a) $C = \{(x, y) \in \mathbb{R}^2 \mid x - 2y = 7\}$.

(b) $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\}$.

(c) $C = \{(x, y) \in \mathbb{R}^2 \mid x = y^2\}$.

(4) [**Marked**] Consider the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^6(1 - x^2)\}.$$

(a) Is C a curve? If yes, why yes? If no, why not?

(b) Find a parametrisation of C and sketch C. Be sure to specify the domain of your parametrisation.

Hint: a nice set of identities to consider are $\cos^2\left(\frac{t}{2}\right) = \frac{1+\cos t}{2}$, $\sin^2\left(\frac{t}{2}\right) = \frac{1-\cos t}{2}$, *and* $\sin t = 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)$.

(c) Find the tangent line to C at $\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)$.

(5) [**Tutorial**] Consider the *ellipse* given by

$$E = \{(x, y) \in \mathbb{R}^2 \mid 3x^2 + 2y^2 = 6\}.$$

(a) Show that E is a curve.

(b) Find a parametrisation of E that passes through the point $(-\sqrt{2}, 0) \in E$.

(c) Find the tangent line to E at $(-\sqrt{2}, 0)$.

(6) [**Tutorial**] Mr Error recently attempted an *Introduction to Differential Geometry* problem sheet and did quite poorly. He definitely needs some help! Here, you can assume the role of a TA for *MTH5113* and help Mr Error see the error of his ways.

(a) Consider the following parabola in $\mathbb{R}^2$:

$$P = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}.$$

The following are two different (correct) parametrisations of P:

$$\gamma : \mathbb{R} \to P, \qquad \gamma(t) = (t, t^2),$$
$$\lambda : \mathbb{R} \to P, \qquad \lambda(u) = (u+1, (u+1)^2).$$

Mr Error computed the tangent line to γ at $t = 0$ and obtained

$$T_\gamma(0) = \left\{ s \cdot (1,0)_{(0,0)} \,\middle|\, s \in \mathbb{R} \right\}.$$

He then computed the tangent line to λ at $u = 0$ and obtained

$$T_\lambda(0) = \left\{ s \cdot (1,2)_{(1,1)} \,\middle|\, s \in \mathbb{R} \right\}.$$

Mr Error noticed that $T_\gamma(0)$ and $T_\lambda(0)$ were not the same, and concluded that tangent lines are not independent of parametrisation! Where did Mr Error err?

(b) Next, Mr Error considered the set

$$C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^4 = 0\}.$$

He noticed that C is a level set of the function $f(x,y) = x^2 + y^4$ and hence deduced that C is curve. Moreover, he observed that C consists of only a single point,

$$C = \{(0,0)\},$$

hence he concludes that a single point at the origin must be a curve (as we defined it in this module)! How did Mr Error go so far astray?

(7) *(Parametrise me!)* Consider the following curves:

$$C_1 = \{(x,y) \in \mathbb{R}^2 \mid x = y^4\},$$
$$C_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ x = 0\},$$
$$C_3 = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1,\ x + y - z = 2\}.$$

(You can assume you already know each of the above is a curve.)

(a) Give one parametrisation of C_1 whose image is all of C_1.

(b) Give one parametrisation of C_2 whose image is all of C_2.

(c) Give one parametrisation of C_3 whose image is all of C_3.

(8) *(Bad function? No problem!)* Find a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$C = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$$

satisfies the following: (i) $\nabla f(x, y)$ vanishes for every $(x, y) \in C$, but (ii) C is a curve.

(9) *(Level set theorem, in 3-d!)* Recall that one can often show that subsets of $\mathbb{R}^2$ are curves by showing that they are "good" level sets of functions. In fact, there is a corresponding result for subsets of $\mathbb{R}^3$, though the statement is a bit more complicated:

> **Theorem.** Let $U \subseteq \mathbb{R}^3$ be open and connected, and let $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ both be smooth functions. Also, let $c_f, c_g \in \mathbb{R}$, and let C be the level set
>
> $$C = \{(x, y, z) \in U \mid f(x, y, z) = c_f,\ g(x, y, z) = c_g\}.$$
>
> If $\nabla f(\mathbf{p}) \times \nabla g(\mathbf{p})$ is nonzero for every $\mathbf{p} \in C$, then C is a curve.

Using this theorem, show that the following subsets of $\mathbb{R}^3$ are curves:

(a) $C_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2,\ y = x\}$.

(b) $C_2 = \{(\cos t, \sin t, t) \mid t \in \mathbb{R}\}$.