§5 Inverse function Theorem

WEEK 4

§5.1 Inverse functions

Defⁿ 5.1.1 Let $f: U \to \mathbb{R}$ and let $V = f(U)$ be the image, then f is invertible if $\exists\ g: V \to U\ (\subseteq \mathbb{R})$ such that

$$g \circ f(x) \overset{\text{DEF}}{=} g(f(x)) = x, \forall x \in U \text{ and } f \circ g(y) = y, \forall y \in V.$$

g is called the inverse function of f and denoted by

~~$\frac{1}{f}$~~ f^{-1}: i.e. $f^{-1}(y) = x$ if ~~$x = g()$~~ $f(x) = y$.

$g(y) = x$.

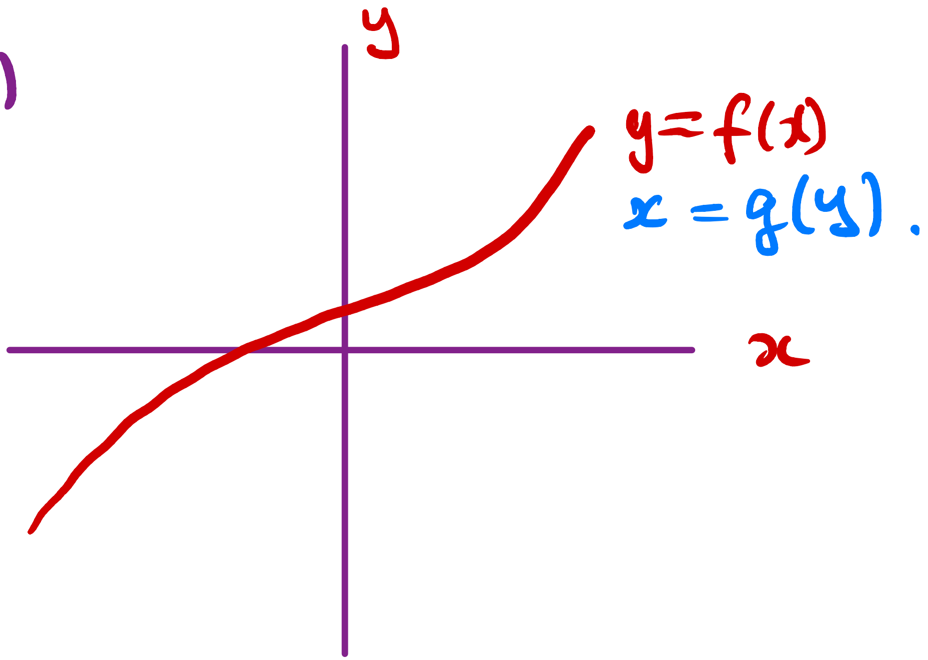
Theorem 5.1.2. (Properties of the inverse)

(i) The inverse is unique. Hence we will write $g = f^{-1}$

(ii) If f is invertible, then f^{-1} $(= g)$ is invertible.

i.e. $(f^{-1})^{-1}(x) = f(x)$

$y = f(x)$



Proof

(i) Let $V = f(U)$ and $g_1, g_2 : V \to \mathbb{R}$ be inverses of function f, then $x = g_1(f(x)) = g_1(y)$, $x = g_2(f(x)) = g_2(y)$. $\underline{(\forall x \in U)}$.

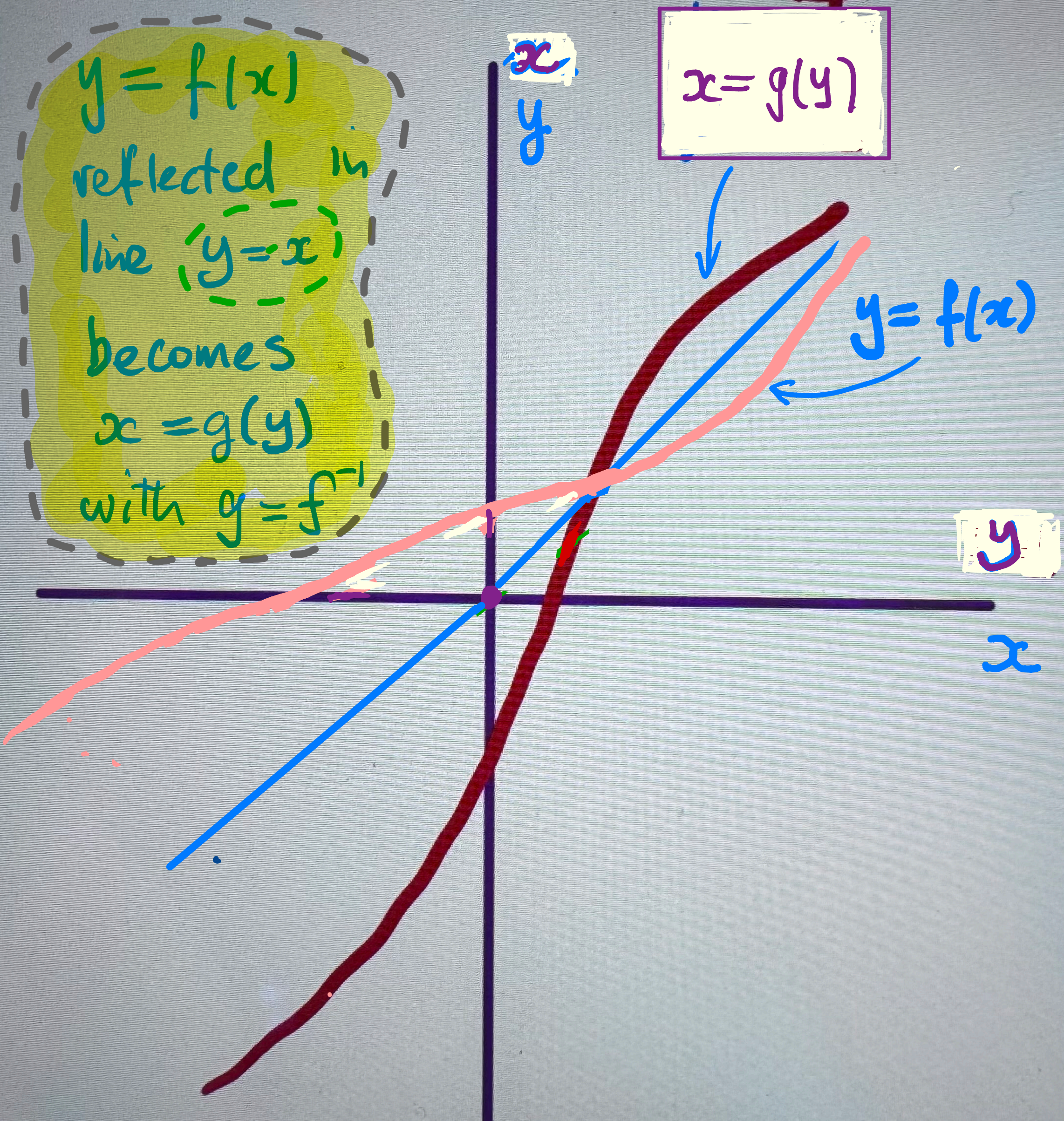
Therefore $g_1(y) = g_2(y)$ $\forall y \in V$, and so $g_1 = g_2$

(ii) Note $g^{-1} = f$: $g^{-1}(x) = y$ given by $x = g(y)$ $(f \circ f^{-1}(y) = f(x) = y)$

Mirror images

$\text{graph}(f) = \{(x, f(x)), x \in U\} = \{(x, y) \mid x \in U, y \in V\}$

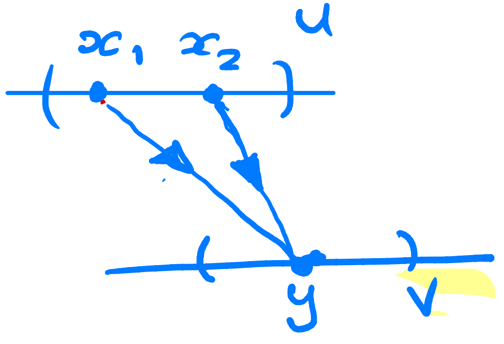
$\text{graph}(g) = \{(y, g(y)), y \in V\} = \{(y, x) \mid y \in V, x \in U\}$



$y = f(x)$ reflected in line $\boxed{y = x}$ becomes $x = g(y)$ with $g = f^{-1}$

Theorem 5.1.3 A function is invertible if and only if f is injective.

Proof Let $f: U \to V$ be invertible, i.e an inverse function exists.



Suppose $f(x_1) = f(x_2)$, then

$$x_1 = f^{-1}(f(x_1)) = f^{-1}(f(x_2)) = x_2$$

$\Rightarrow$ f is injective.

Let f be injective and $V = f(U)$ $\Rightarrow$ that if $y \in V = f(U)$, then $\exists$ unique $x \in U$, such that $f(x) = y$. Define $f^{-1}(y) = x$, this is well-defined.

If $g(y) = x_1$ and $g(y) = x_2$, then $f(x_1) = f(x_2) = y$

$\Rightarrow x_1 = x_2$ (by injectivity). $\square$

Examples

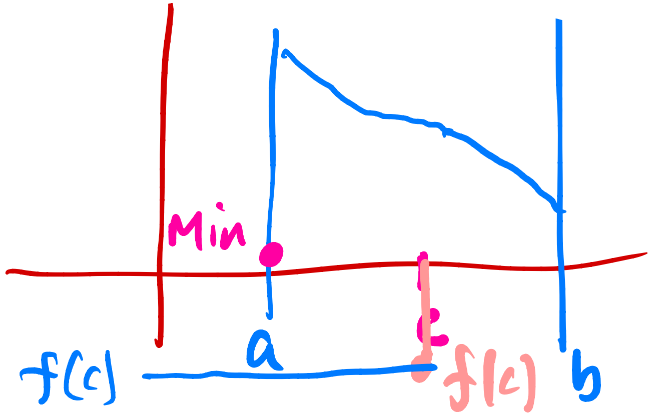
$f: \mathbb{R} \to \mathbb{R}$; $f(\mathbb{R}) = \mathbb{R}^+ \cup \{0\}$
$x \mapsto x^2$

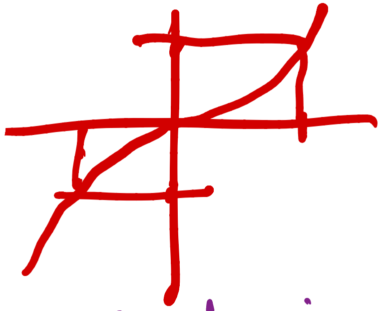
f does not have an inverse

$g: f(\mathbb{R}) \to \mathbb{R}$

because f is non-injective

e.g. $f(-3) = f(3) = 9$ so,
def^n g $f^{-1}(9)$ would be ambiguous.





However, partial inverses can be constructed

$g: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$

where $g(y) = +\sqrt{y}$

$(g \circ f)(x) = g(f(x)) =$

$= g(x) = +\sqrt{x^2} = x$, for $x > 0$.

Also $f: \mathbb{R}^- \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$
$x \mapsto x^2 = y$

$$\boxed{f^{-1}(y) = -\sqrt{y}}$$

Theorem 5.1.4. Let $f: [a,b] \to \mathbb{R}$ be continuous and injective. Then f attains its minimum or maximum at a or b.

Proof Assume $f(a) < f(b)$.

Suppose $\exists\, c$, $a < c < b$, such that $f(c) < f(a) < f(b)$

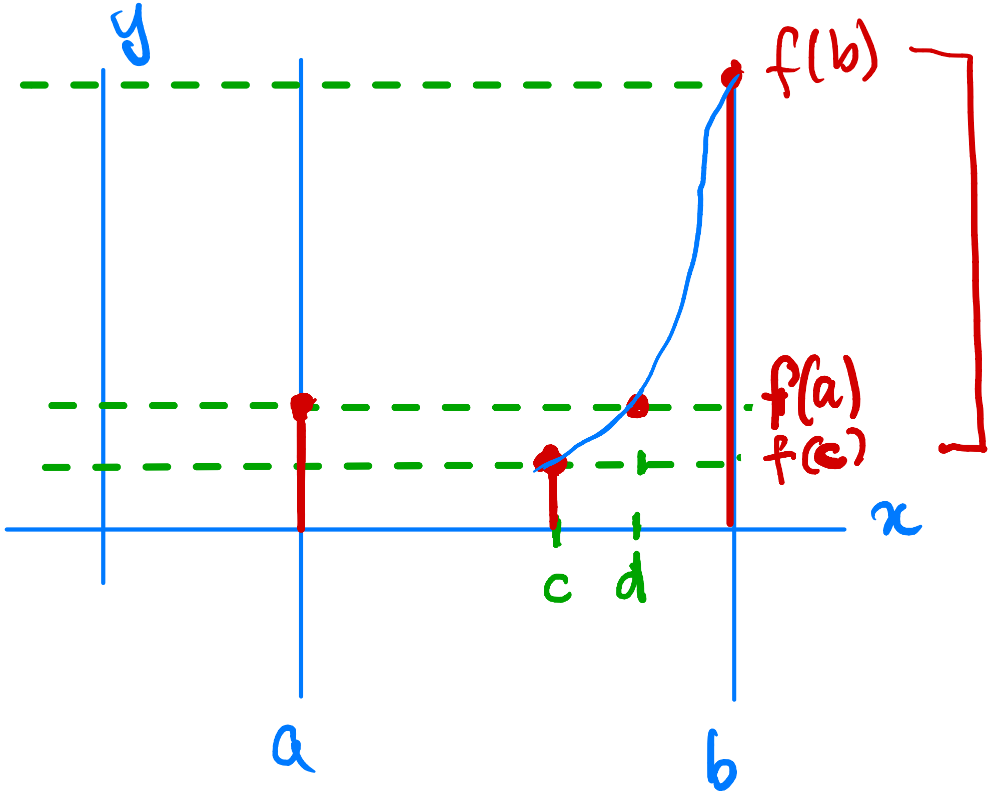
By The intermediate value theorem (IVT) $\exists\, d$ such that $f(d) = f(a)$ & $d \neq a$ because

$$a < c < d < b$$

$\therefore$ f is not injective $\otimes$ contradiction

$$\therefore f(c) \not< f(a).$$

Similarly $f(c) \not> f(b)$.



f is continuous on $[a,b]$

and $\therefore$ on $[c,b]$

Assume $\exists\ c \in (a,b)$ s.t.

$f(c) < f(a) < f(b)$

$\therefore$ IVT gives a $d \in (c,b]$

such that $f(d) = f(a)$

and $a \neq d$ $(a < c < d)$

$\therefore$ f is not injective $\otimes$

(contrary to assumption)

__Note__ assume $f(c) > f(b)$ and show $\exists\ e$ s.t.

$a < e < c < b$ with $f(e) = f(b)$ and $e \neq b$.

Trivial observation ~

consider $f(x) = mx + C$, let $y = f(x)$, then

$y = mx + c \Rightarrow x = \frac{y-c}{m}$, for $m \neq 0$ $\left(\text{note } m = f'(x)\right)$

So inverse exists for $m \neq 0$, and if $g(y) = \frac{y-c}{m}$

$$g'(y) = \frac{1}{m} = \frac{1}{f'(x)}$$

We prove a result like this for functions f with non-zero derivative.

§5.2 Inverse function theorem

Note a linear function $y = mx + c$ can be "inverted" to give $x = (y-c)/m$, provided $m \neq 0$ and "$m = \frac{dy}{dx}$".

Let f be an injective function on an open interval I.

We can show that:

f is strictly increasing or decreasing on I, &

The image is an interval J, $I, J \subseteq \mathbb{R}$

$$f^{-1} \circ f(x) = x \quad \forall x \in I, \text{ and}$$

$$f \circ f^{-1}(y) = y \quad \forall j \in J$$

Prod of numbers

$g'(f(x)).f'(x),$
$= (g \circ f)'(x)$

comp of fns

Differentiating with respect to x gives

$$g'(y) \cdot f'(x) = (f^{-1})'(f(x)) \cdot f'(x) = \frac{d}{dx}(x) = 1 \quad \forall\, x \in I$$

"IF f^{-1} and f ARE DIFFERENTIABLE"

If $x_0 \in I$, then

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}, \quad f'(x_0) \neq 0$$

We assumed f^{-1} is differentiable here, can we prove it?

Theorem 5.2.1 (Inverse function Theorem)

Let f be an injective continuous function on an open interval I and let $J = f(I)$. If f is differentiable at $x_0 \in I$, and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof Note I is an open interval and we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \neq 0$$

$$\therefore \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} \neq 0.$$

$x = f^{-1}(y)$, $x_0 = f^{-1}(y_0)$ $\circledast$

$$\Rightarrow \lim_{x \to x_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

But f is differentiable and therefore continuous

so as $x \to x_0$, $y \to y_0$

$$\circledast = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = (f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}$$

$\therefore$ $g = f^{-1}$ is differentiable at $y = y_0$, $g'(y_0)$ exists

Note $g \circ f(x) = g(y) = x$ and $f \circ g(y) = f(x) = y$

$$\therefore f^{-1} = g \, , \; g^{-1} = f$$

<u>Corollary</u> If $f : U \to \mathbb{R}$ is strictly increasing (decreasin

f · invertible

<u>Corollary</u> If $f : U \to \mathbb{R}$ is strictly increasing (decreasing), then f is invertible.

$|x+1| = x+1 \quad x+1>0$

$= -(x+1) \quad x+1<0 \quad f(x) = |x+1|$

$x = 0.$

Example 5.2.2.

(i) Let $f(x) = +x^{1/2}$, $x \in (0, \infty)$

Then f is injective (why?), continuous and differentiable $\forall\, x \in (0,\infty)$. Check!

$$f'(x) = \frac{1}{2x^{1/2}} \; (\neq 0, \text{ for } x > 0).$$

Also $f'(x) > 0$, and so $f(x)$ is strictly increasing on $(0,\infty)$.

Note

differentiable

$\Downarrow \; \cancel{\Uparrow}$

continuous

By The IVT, for $y_0 = f(x_0)$, $y_0 = x_0^{1/2}$

$$(f^{-1})'(y) = \frac{1}{1/2x_0^{1/2}} = 2x_0^{1/2} = 2y_0$$

$g(y) = x = y^2$

$g'(y) = 2y$

$y_0 = x_0^{1/2}$

$x_0 = y_0^2$

Note: $f^{-1}(y_0) = y_0^2$ and $(f^{-1})'(y_0) = 2y_0$, as expected!

$$\boxed{(g\circ f)'(x) = g'(\underset{y}{f(x)})\, f'(x)}$$ ✓✓

$g = f^{-1}$ $\quad (g\circ f)(x) = x$

Let $y = f(x)$, $\quad x = g(y)$, $\quad g = f^{-1}$.

$$g'(y)\cdot f'(x) = 1$$

$$\boxed{g'(y) = \frac{1}{f'(x)}}$$

(ii) Let $f(x) = e^x \; (> 0)$. $\therefore \; f'(x) = e^x > 0 \Rightarrow$

f is differentiable, injective and strictly increasing.

By IVT $\quad (f^{-1})'(y) = \dfrac{1}{f'(x)} = \dfrac{1}{e^x} = \dfrac{1}{y}$.

Note $f^{-1}(y) = \ln(y)$, $\quad (f^{-1})'(y) = \dfrac{1}{y}$, as expected!

$y = f(x)$

$x = g(y)$

$g'(y) = \dfrac{dx}{dy}$

(iii) Consider $f(x) = \dfrac{1}{1-x}$, $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$.

Use Inv. Fn. Thm:

$$(f^{-1})'(y) = 1/f'(x) = 1/\left[-\frac{(-1)}{(1-x)^2}\right] = (1-x)^2 = \frac{1}{y^2}$$

$$\therefore \; (f^{-1})'(y) = \frac{1}{y^2}$$

CHK: $y = \dfrac{1}{1-x}$, $\Rightarrow x = 1 - \dfrac{1}{y}$

$\dfrac{dx}{dy} = \dfrac{1}{y^2}$ ✓

(iv) $f(x)=\sin(x)$, $x\in(-\frac{\pi}{2},\frac{\pi}{2})$

By IVT: $(f^{-1})'(y) = \frac{1}{\sqrt{1-y^2}}$

C+IK: ?

Note $f^{-1}(y)=\arcsin(y)$

$f(x) = \sin(x)$

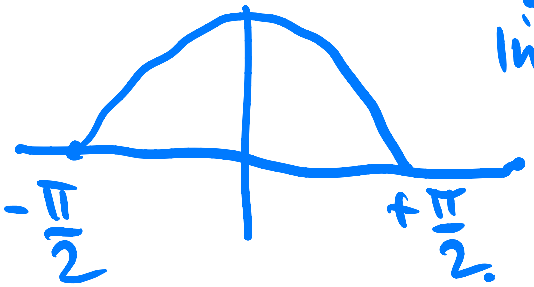
$x = g(y)$

$f'(x) = \cos(x) > 0$

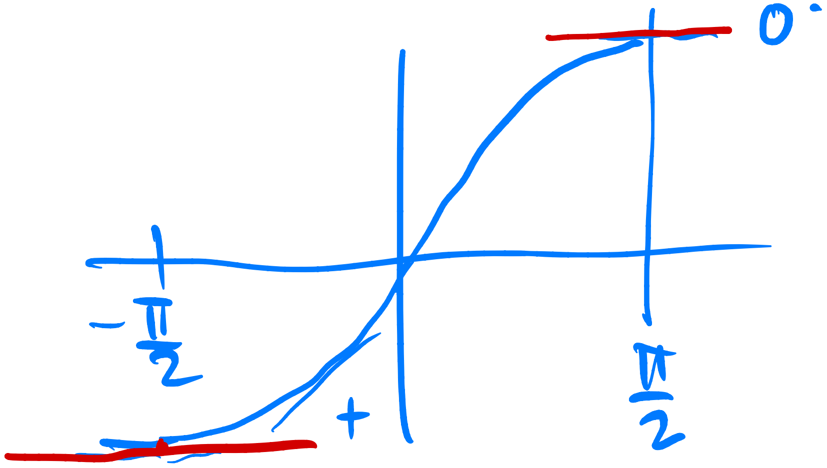
strictly increasing fn,

$g = f^{-1} = \underline{\arcsin}\ (=(\sin^{-1})(y))$

g f $=\underline{\sin}$



END OF WEEK 4



$f'(\frac{\pi}{2}) = 0 \quad , f''(\frac{\pi}{2}) = -1$

$f'(x) = \cos x$

$f''(x) = -\sin x.$

$f''(\frac{\pi}{2}) =$

$\boxed{(\sin^{-1}) \circ \sin}(x) = \boxed{x}$

$(\sin^{-1})'(\sin x) \cdot \sin'(x) = 1$

Let $y = \sin x$ ✓✓

$\arcsin'(y) \cdot \sin'(x) = 1$

$\arcsin'(y) = \frac{1}{\cos x} = \frac{1}{+\sqrt{1-\sin^2 x}}$

> 0

$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Exercises Investigate differentiability of

$$f:\mathbb{R}\to\mathbb{R} \qquad x \longmapsto (x+1)|x|$$

$$g:\mathbb{R}\to\mathbb{R} \qquad x \longmapsto (x-1)|x-1|$$

Note: $|x| = \begin{cases} x & | \quad x \geq 0 \\ -x & | \quad -x \leq 0 \end{cases}$

$$\therefore \quad f(x) = \begin{cases} (x+1)\,x\,, & x > 0 \\ -(x+1)\,x\,, & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x+1\,, & x > 0 \\ -2x-1\,, & x < 0 \end{cases}$$

Note $f'(x) \longrightarrow 1$ as $x \to 0^+$

$f'(x) \longrightarrow -1$ as $x \to 0^-$

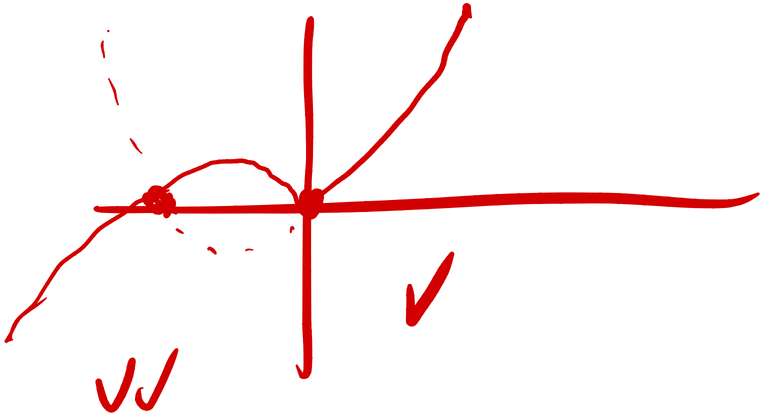
$$\therefore f'(0) \nexists$$

$$g(x) = \begin{cases} (x-1)(x-1)\,, & x > 1 \\ -(x-1)(x-1)\,, & x < 1 \end{cases} \Rightarrow g'(x) = \begin{cases} 2x-2\,, & x > 1 \\ -2x+2 & x < 1 \end{cases}$$

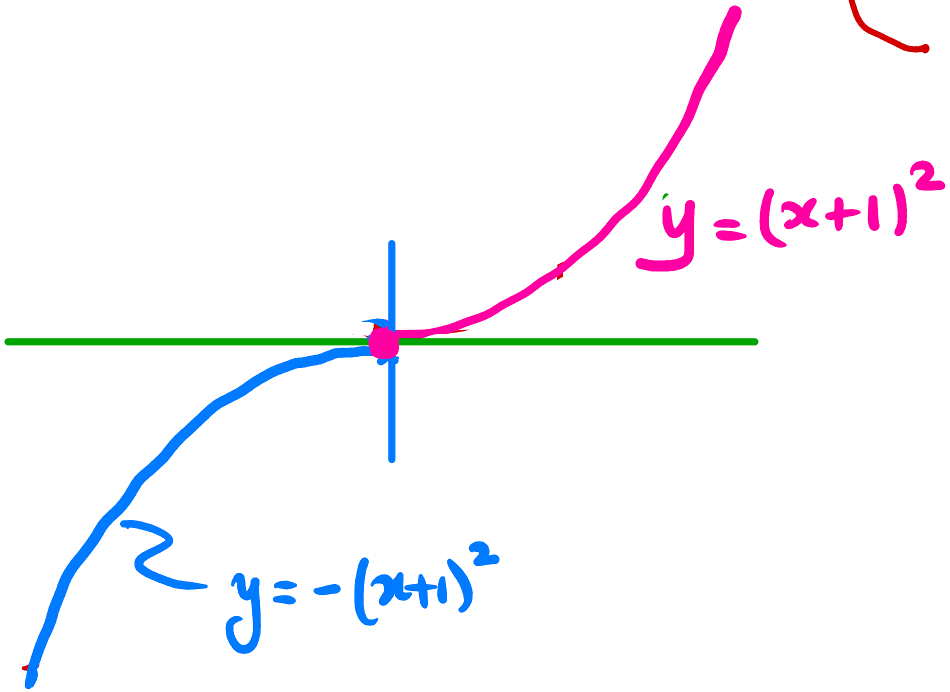
$g'(1^-) = 0$, $g'(1^+) = 0$ $\therefore g'(0) = 0.$

$x^2+x \quad x>0$ ✓

$-x^2-x \quad x<0$ ✓✓



$$g(x) = (x+1)|x+1| = \begin{cases} (x+1)^2 \; * & x+1>0 \; , \; x>-1 \\ -(x+1)^2 \; * & x+1<0 \quad x<-1 \end{cases}$$



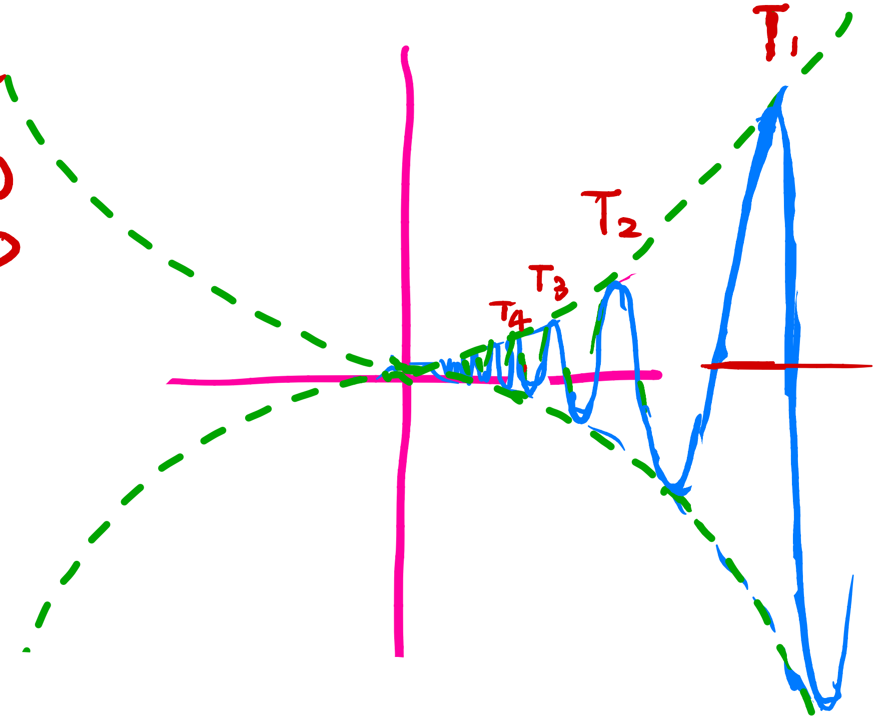
$f'(-1^-)=0$

$f'(-1^+)=0$

$\therefore f'(-1)=0$

Note some function can be differentiable but the derivative need not be differentiable

Consider $f(x)=\begin{cases} x^2\sin(1/x), & x\neq 0 \\ 0, & x=0 \end{cases}$



$f'(0)=0$ (see earlier)

$f'(x)=2x\sin\left(\frac{1}{x}\right)+x^2\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$

$=2x\sin\left(\frac{1}{x}\right)+\cos\left(\frac{1}{x}\right)$

The term $\cos\left(\frac{1}{x}\right)$ oscillates infinitely often between 1 & -1 as $x\to 0$ whereas $\left|2x\sin\left(\frac{1}{x}\right)\right|$ is bounded by $2|x|$ so for $2|x|$ say $<\frac{1}{2}$ $f'(x)$ oscillates infinitely between $>\frac{1}{2}$ and $<-\frac{1}{2}$ as $x\to 0$

$\therefore$ f' not continuous.

COURSEWORK

4. 1 [redacted]

1) Let the function $f\colon (0,\pi) \to \mathbb{R}$ be given by $x \mapsto \cos(x)$. Show that f is invertible and that the inverse $g(y) = f^{-1}(y)$ is differentiable. Find the derivative g'.

Compute the Taylor polynomial $T_{1,0}(y)$ about zero of degree one for g. What is the remainder term in the Lagrange form?

Hence show that for $|y| \leq 1/2$

$$|g(y) - \pi/2 + y| \leq \sqrt{3}/18 \approx 0.096 \,.$$

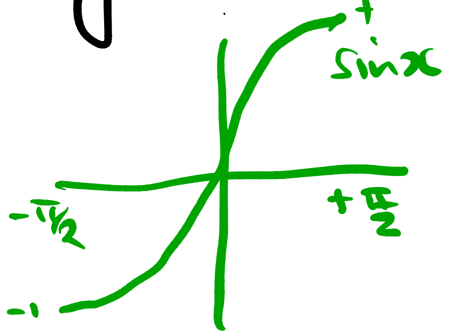
$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \cos(x), \quad f'(x) = -\sin(x), \quad x \in (0,\pi)$

$x \mapsto \cos(x)$

$\sin(x) > 0$ for $x \in (0,\pi)$ $\therefore$ f is strictly decreasing & continuous and so f is invertible on its co-domain

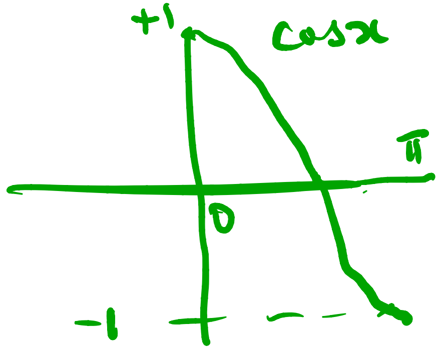
$$f\left((0,\pi)\right) = (\underset{\overset{\shortparallel}{v}}{-1}, 1)$$

with $\overset{\shortparallel}{u}$ under $(0,\pi)$



$g = f^{-1}$ exists as $g : (-1,1) \to (0,\pi)$

$g \circ f(x) = x \;, \forall\; x \in (0,\pi)$



$$g'(f(x))\,(f'(x)) = 1 = g'(y)\cdot f'(x)$$

Let $y \in (-1,1)$, $g'(y)\cdot f'(x) = 1 \Rightarrow g'(y) = 1/f'(x)$

i.e. $g'(y) = 1/(-\sin x) = -\dfrac{1}{\sqrt{1-\cos^2(x)}} = \dfrac{-1}{\sqrt{1-y^2}}$

$g(0) = \pi/2$
$f(\pi/2) = 0$

$$T_{(1,0)}f(x) = 1 + -\frac{\sin(0)}{1}x = 1 + 0.x = 1$$

$$R_{(1,0)}f(x) = \frac{x^2}{2!}f^{(2)}(c) = \frac{x^2}{2!}(-\sin(c)),\quad c \in (0,x)$$

$$g(y) = T_{(1,0)}g(y) = g(0) + \frac{y}{1!}g'(0) = \frac{\pi}{2} + y(-1) + R$$

$$\left.\frac{-1}{\sqrt{1-y^2}}\right|_{y=0} = -1$$

$|y| < \frac{1}{2}$

$$R_{(1,0)}g(y) = \frac{y^2}{2!}g''(c),\ 0 < c < x$$

$$g'(c) = \frac{-1}{\sqrt{1-y^2}},\ g''(c) = (-1)\left(-\tfrac{1}{2}\right)(1-y^2)^{-3/2}\cdot(-2y)\Big|_c,\quad 0<c<y$$

i.e. $g''(c) = \left(\frac{1}{2}\right)\frac{1}{(1-y^2)^{3/2}}(-2y)\Big|_c =$

$$\therefore \left|R_{(1,0)}\,g(y)\right| = \frac{y^2}{2}\,\frac{c}{(1-c^2)^{3/2}}$$

$|y| < \frac{1}{2}$

$\therefore |c| < \frac{1}{2}$ ✶.

$$< \frac{\left(\frac{1}{2}\right)^2}{2}\,\frac{\left(\frac{1}{2}\right)}{\left(1-\left(\frac{1}{2}\right)^2\right)^{3/2}}$$

$$= \frac{1}{16}\,\frac{1}{\left(\frac{3}{4}\right)^{3/2}} = \frac{1\cdot 8}{16\cdot 3\sqrt{3}} = \frac{1}{6\sqrt{3}}$$

If $c^2 < \frac{1}{4}$

$\Rightarrow -\frac{1}{4} < -c^2$

$\Rightarrow 1-\frac{1}{4} < 1-c^2$

$\frac{1}{1-c^2} < \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

$$\therefore \left|g(y) - \frac{\pi}{2} + y\right| = \left|R_{(1,0)}\,g(y)\right| < \frac{1}{6\sqrt{3}} = \frac{\sqrt{3}}{18}$$

COURSEWORK 4.2

2) Let $f : (-1, \infty) \to \mathbb{R}$, $x \mapsto \sin(\pi\sqrt{1+x})$. Show that

$$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0\,.$$

Show that for all $n \in \mathbb{N}$

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0\,.$$

Hence find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\pi\sqrt{1+x})$.
Hint: If you wish you may use Leibniz's formula for the derivative of a product of n-times differentiable functions g and h, $(gh)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)} h^{(k)}$.

$$f(x) = \sin(\pi\sqrt{1+x})$$

$$f^{(1)}(x) = \cos(\pi\sqrt{1+x}) \cdot \frac{\pi}{2} \cdot \frac{1}{\sqrt{1+x}}$$

$$f^{(2)}(x) = -\frac{\pi^2}{4} \sin(\pi\sqrt{1+x}) \underset{\underset{\frac{1}{(1+x)}}{\shortparallel}}{\frac{1}{(\sqrt{1+x})^2}} + \cos(\pi\sqrt{1+x})\left(-\frac{\pi}{4}(1+x)^{-3/2}\right)$$

$$4(1+x)f^{(2)}(x) = -\pi^2\sin(\pi\sqrt{1+x}) + \cos(\pi\sqrt{1+x})\left(-\pi\frac{1}{\sqrt{1+x}}\right)$$

$$\| \qquad\qquad \| \qquad\qquad \|$$

$$-\pi^2 f^{(0)}(x) \qquad\qquad -2f^{(1)}(x)$$

$$4(1+x)f^{(2)}(x) + \underline{\overset{\textcircled{2}}{2\cdot f^{(1)}(x)}} + \pi^2 f(x) \qquad n=0 \qquad \text{as req}^{d}$$

Diff:

$$4(1+x)f^{(3)}(x) + \underline{4f^{(2)}(x) \overset{\textcircled{6}}{+} 2f^{(2)}(x)} + \pi^2 f^{(1)}(x) = 0 \qquad n=1$$

$$4(1+x)f^{(4)} + \underline{4f^{(3)}(x) \overset{\textcircled{10}}{+} 4f^{(3)}(x) + 2f^{(3)}(x)} + \pi^2 f^{(2)}(x) = 0 \qquad n=2$$

For general n:

$$4(1+x)f^{(n+2)}(x) + 2(2n+1)f^{(n+1)}(x) + \pi^2 f^{(n)}(x) = 0$$

Evaluate $f(0)$ & $f^{(1)}(0)$, then use recursion

$$4f^{(n+2)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0.$$

<u>Exercises</u> Investigate differentiability of

$$f : \mathbb{R} \to \mathbb{R} \qquad x \longmapsto (x+1)|x|$$

$$g : \mathbb{R} \to \mathbb{R} \qquad x \longmapsto (x-1)|x-1|$$

Note: $|x| = \begin{cases} x & | \; x \geq 0 \\ -x & | \; -x \leq 0 \end{cases}$ $\therefore$ $f(x) = \begin{cases} (x+1)\,x \, , & x > 0 \\ -(x+1)\,x \, , & x < 0 \end{cases}$

$$f'(x) = \begin{cases} 2x+1 \, , & x > 0 \\ -2x-1 \, , & x < 0 \end{cases}$$

Note $f'(x) \longrightarrow 1$ as $x \to 0^+$

$f'(x) \longrightarrow -1$ as $x \to 0^-$

$$\therefore f'(0) \nexists$$

$$g(x) = \begin{cases} (x-1)(x-1) \, , & x > 1 \\ -(x-1)(x-1) \, , & x < 1 \end{cases} \Rightarrow g'(x) = \begin{cases} 2x-2 \, , & x > 1 \\ -2x+2 \, , & x < 1 \end{cases}$$

$$g'(1^-) = 0 \, , \quad g'(1^+) = 0 \quad \therefore g'(0) = 0.$$