MTH6105 – Algorithmic Graph Theory **Spring 2024**

Problem Sheet 3 **F. Fischer**

You are expected to **attempt all exercises** before the seminar and to **actively participate** in the seminar itself.

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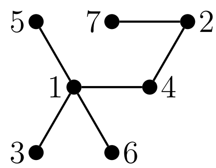
1. Show that a graph G is a tree if and only if it contains no loops and a unique u–v-path for every $u, v \in V(G)$.
   (a) For the direction from right to left, consider a graph G without loops and with unique u–v-paths, and show that G is connected and acyclic.
   (b) For the direction from left to right, show that if T is a tree, then it contains unique u–v-paths.

2. (a) Give the Prüfer code of the following tree.



   (b) Draw the tree with Prüfer code $(1, 2, 3, 2, 1)$

3. For $n \in \{0, 1, 2, 3, \dots\}$, let Q_n be the simple graph with

$$V(Q_n) = \{X : X \subseteq [n]\},$$
$$E(Q_n) = \{XY : X, Y \in V(Q_n),\ |(X \setminus Y) \cup (Y \setminus X)| = 1\}.$$

   (a) Draw Q_0, Q_1, Q_2, and Q_3.
   (b) Determine $d_{Q_{13}}(\{1, 3\})$.
   (c) Give all values of n for which Q_n is a tree. Justify your answer.
   (d) Show that Q_n is connected for all n. You may want to consider $X, Y \in V(Q_n)$ such that $|(X \setminus Y) \cup (Y \setminus X)| = k$, and show existence of an X–Y-path by induction on k.