Mathematical Tools for Asset Management
MTH6113

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<u>Practice Session 3</u>

1. An investor has a utility of wealth function given by: $U(w) = w + kw^2$ where k is a constant and $w > 0$ is the investor's wealth.
   (i) Show under what circumstances this utility function would be appropriate for a risk-averse investor who prefers more wealth to less

Prefers more to less: $U'(w) > 0$

Risk averse: $U''(w) < 0$

$$U(w) = w + kw^2$$

$$U'(w) = 1 + 2kw > 0$$

$$U''(w) = 2k < 0$$

We need to restrict the wealth values if $k < 0$:

$1 + 2kw > 0$ means $0 < w < \frac{1}{-2k}$

Show whether the function exhibits:
a. increasing, decreasing or constant absolute risk aversion

Solution

$$A(w) = -\frac{U''(w)}{U'(w)} = -\frac{2k}{1+2kw}$$

$$A'(w) = \frac{2k}{(1+2kw)^2}(2k) = \frac{4k^2}{(1+2kw)^2}$$

$A'(w) > 0$ if $k \neq 0$ increasing ARA
$A'(w) = 0$ if $k = 0$ constant ARA

b. increasing, decreasing or constant relative risk aversion

$$R(w) = -w\frac{U''(w)}{U'(w)} = -\frac{2kw}{1+2kw}$$

$$R'(w) = -\frac{2k}{1+2kw} + \frac{2kw}{(1+2kw)^2}(2k) = \frac{-2k-4k^2w+4k^2w}{(1+2kw)^2} = \frac{-2k}{(1+2kw)^2}$$

$R'(w) < 0$ if $k > 0$ declining RRA

$R'(w) = 0$ if $-2k = 0$ or $k = 0$ constant RRA

$R'(w) > 0$ if $k < 0$ increasing RRA

2. The loss to an individual from the occurrence of an adverse event is £200 and the probability that the event will occur is 0.2. Insurance is available against the loss.

(i) The utility of any level of wealth for a particular individual is given by the function: $U = 200 + w + \frac{500}{w}$ where U is the utility and w is the wealth of the individual. The individual's initial level of wealth is £250.

Calculate, to the nearest penny, the maximum premium the individual is prepared to pay to cover this risk.

Answer

Maximum premium is c_x where $U(250 - c_x) = EU[250 + X]$ where X represents the potential loss; X belongs to {0,-200} with respective probabilities {0.8, 0.2}

$$200 + (250 - c_x) + \frac{500}{(250 - c_x)} = 0.8\left(200 + (250 + 0) + \frac{500}{250 + 0}\right) +$$
$$+0.2\left(200 + (250 - 200) + \frac{500}{250 - 200}\right)$$

Which simplifies to

$$(450 - c_x)(250 - c_x) + 500 = 413.6 \times (250 - c_x)$$

$c_x = 38.77$ or 247.63 (take first value as the second premium is greater than max loss)

The utility of any level of capital (can be regarded as insurer's wealth) for a particular insurer is given by the function: $U = 5{,}000 + 0.7\,c$
where U is the utility and c is the capital of the insurer. The insurer's initial level of capital is £1,500.

Calculate the minimum premium the insurer is prepared to charge for taking on this risk.

Answer

The choice faced by the insurer is whether to offer insurance for a premium (take gamble) or not offer insurance at all (not take gamble).

Minimum premium is Q which is received in all possible states (not random). The loss is the same as before and is random.

$$EU[1500 + Q + X] = U(1500)$$

$$0.2\big(5000 + 0.7(1500 + Q - 200)\big) + 0.8\big(5000 + 0.7(1500 + Q)\big) = 5000 + 0.7 \times 1500$$

$$\Rightarrow 6050 = 6050 + Q - 40$$

$$\Rightarrow Q = 40$$

2. Consider the two risky assets A and B with cumulative probability distribution functions:

$$F_A(x) = x$$
$$F_B(x) = \sqrt{x}$$

In both cases $0 \leq x \leq 1$.

1. Show that A is preferred to B on the basis of first-order stochastic dominance.

Check first that we have well defined cdfs:

$F_A(0) = 0, \qquad F_A(1) = 1$

$F_B(0) = 0, \qquad F_B(1) = 1$

A is preferred to B on the basis of first-order stochastic dominance if:

$F_A(x) \leq F_B(x)$ for all $0 \leq x \leq 1$ and

$F_A(x) < F_B(x)$ for some value of x in this range

i.e. if $x \leq x^{1/2} \Leftrightarrow x - x^{1/2} \leq 0 \Leftrightarrow x^{1/2}\left(x^{1/2} - 1\right) \leq 0$

This clearly holds for all $0 \leq x \leq 1$ the equality being strict for $0 < x < 1$.

Hence A first-order dominates B.

2. Verify explicitly that A also dominates B on the basis of second-order stochastic dominance

A second-order dominates B if :

$$\int_0^x F_A(y)dy \leq \int_0^x F_B(y)dy$$
$$\text{for all } 0 \leq x \leq 1$$

with the strict inequality holding for some value of x. This means:

$$\frac{1}{2}x^2 - \frac{2}{3}x^{3/2} \leq 0$$

$$\frac{2}{3}x^{\frac{3}{2}}(\frac{3}{4}x^{\frac{1}{2}} - 1) \leq 0$$

This clearly holds for all $0 \leq x \leq 1$ the equality being strict for $0 < x < 1$.

Hence A second-order dominates B.