MTH6127 **Metric Spaces and Topology**

Course work 2 **9 February 2024**

1. For the metric $d_{L^1}(f,g)$ defined by

$$d_{L^1}(f,g) = \int_a^b |f(x) - g(x)|dx,$$

where $f, g \in C[a,b]$, compute the distance $d_{L^1}(f,g)$ between $f(x) = e^x$ and $g(x) = 2$ where $[a,b] = [0,5]$.

We have $d_{L^1}(f,g) = \int_a^b |f(x) - g(x)|dx = \int_0^5 |e^x - 2|dx$. Since $e^x - 2$ changes its sign at $x = \ln 2$ (where $e^x = 2$), we obtain

$d_{L^1}(f,g) = \int_0^{\ln 2}(2 - e^x)dx + \int_{\ln 2}^5 (e^x - 2)dx$
$= (2x - e^x)|_0^{\ln 2} + (e^x - 2x)|_{\ln 2}^5$
$= (2\ln 2 - 2) - (0 - 1) + (e^5 - 10) - (2 - 2\ln 2)$
$= e^5 + 4\ln 2 - 13.$

2. Let $X = \mathbb{R}^m$. For any $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m) \in X$, we set

$$d_\infty(x,y) := \max_k \{|x_k - y_k|\}.$$

Prove that d_∞ defines a metric on X.

(M1) and (M2) are obvious and to check (M3) we note that for $x, y, z \in \mathbb{R}^m$ one has

$$\begin{aligned} d_\infty(x,y) &= \max_{1\le k\le n} |x_k - y_k| = \max_{1\le k\le n} |x_k - z_k + z_k - y_k| \\ &\le \max_{1\le k\le n} (|x_k - z_k| + |z_k - y_k|) \\ &\le \max_{1\le k\le n} |x_k - z_k| + \max_{1\le k\le n} |z_k - y_k| \\ &= d_\infty(x,z) + d_\infty(z,y). \end{aligned}$$

3. Let (X,d) be a metric space. Define two new functions d_a and d_b on $X \times X$ by

$$d_a(x,y) := \min\{d(x,y), 1\}, \quad d_b(x,y) := \frac{d(x,y)}{1 + d(x,y)}, \quad \text{for} \quad x, y \in X.$$

Prove that d_a and d_b are also metrics on X.

Using (M3) for d, we find

$$\begin{aligned} d_a(x,y) &= \min\{d(x,y),1\} \\ &\le \min\{d(x,z)+d(z,y),1\} \\ &\le \min\{d(x,z),1\}+\min\{d(z,y),1\} \\ &= d_a(x,z)+d_a(z,y). \end{aligned}$$

In the inequality of third line one considers the cases (a) $d(x,z) \le 1$ and $d(z,y) \le 1$ and (b) when one of these numbers is greater than 1.

Next we observe that the function $f(x) = \frac{x}{1+x} = 1 - \frac{1}{1+x}$ is monotonically increasing. We therefore find

$$\begin{aligned} d_b(x,y) &= f(d(x,y)) \le f(d(x,z)+d(z,y)) \\ &= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\ &\le \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= d_b(x,z)+d_b(z,y). \end{aligned}$$

4. We define *"the Jungle metric"* d_J on $X = \mathbb{R}^2$ by

$$d_J(x,y) := \begin{cases} |x_2 - y_2| & \text{if } x_1 = y_1, \\ |x_2| + |x_1 - y_1| + |y_2| & \text{otherwise.} \end{cases}$$

("climb down from the tree, walk to another one, climb up the tree"). Prove that d_J defines a metric on X.

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Clearly, if $x = y$ then $d_J(x,y) = 0$. Conversely, if $d_J(x,y) = 0$ then $x_1 = y_1$ and $x_2 = y_2$, i.e. $x = y$. This proves (M1). The axiom (M2) is obvious.

To check (M3) consider $x, y, z \in \mathbb{R}^2$ and assume first that $x_1 \ne y_1$, $y_1 \ne z_1$, $x_1 \ne z_1$. Then

$$\begin{aligned} d_J(x,y) &= |x_2| + |y_2| + |x_1 - y_1| \\ &\le |x_2| + |y_2| + |x_1 - z_1| + |z_1 - y_1| + 2|z_2| \\ &= d_J(x,z) + d_J(z,y). \end{aligned}$$

If $x_1 = y_1 = z_1$ then

$d_J(x, y) = |x_2 - y_2| \leq |x_2 - z_2| + |z_2 - y_2| = d_J(x, y) + d_J(y, z)$.

In the case $x_1 = y_1$ and $z_1 \neq x_1$, we have

$d_J(x, y) = |x_2 - y_2| \leq |x_2| + |y_2|$
$\leq |x_2| + |y_2| + |x_1 - z_1| + |y_1 - z_1| + 2|z_2|$
$= d_J(x, z) + d_J(z, y)$.

Finally in the remaining case $z_1 = x_1$ and $x_1 \neq y_1$ we have

$d(x, y) = |x_2| + |y_2| + |x_1 - y_1|$
$\leq |x_2 - z_2| + |z_2| + |y_2| + |z_1 - y_1|$
$= d_J(x, z) + d_J(z, y)$.