Lecture 3(a) Geometry of linear programming

Aim: to solve LPs with two variables by drawing

Recap quiz

$$(3, -2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5$$

$3x_1 - 2x_2 = 5$ is line in $\mathbb{R}^2$.

Find a vector perpendicular to this line $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Geometrically what happens to the line if we replace 5 with 6? It remains parallel but moves in the direction $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Given an LP, what is a

(i) feasible solution?
Any assignment of values to the variables that satisfies all constraints and sign restrictions,

(ii) optimal solution?
Any feasible solution that achieves the goal i.e. maximise/minimise objective function.

Example

Consider following LP

$$\text{maximise } x_1 + 2x_2$$

subject to

$$x_2 \leq 5 \quad (1)$$
$$x_1 + x_2 \leq 8 \quad (2)$$
$$2x_1 + \tfrac{1}{2}x_2 \leq 12 \quad (3)$$
$$x_1, x_2 \geq 0 \quad (4)$$

e.g. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ feasible

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ not feasible

(3) violated

$2 \times 6 + \frac{1}{2} \times 2 = 13 \not\leq 12$

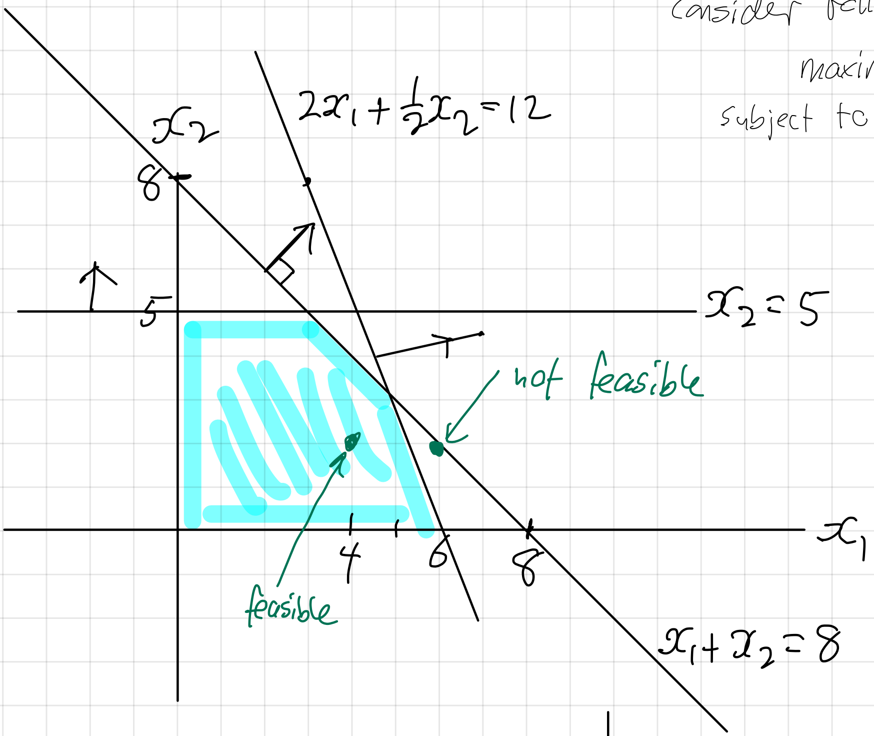
(i) Sketch the region of all feasible solutions (called the <u>feasible region</u>)

(ii) Use this to find an optimal solution.

Consider following LP

maximise $x_1 + 2x_2$

subject to
$x_2 \leq 5$ ①
$x_1 + x_2 \leq 8$ ②
$2x_1 + \frac{1}{2}x_2 \leq 12$ ③
$x_1, x_2 \geq 0$ ④



② $x_1 + x_2 = 8$

i.e. $(1, 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 8$

line $\perp$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

passes through $\begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 8 \end{pmatrix}$

Every point below the line $x_1 + x_2 = 8$ (i.e. other side of normal) satisfies $x_1 + x_2 \leq 8$

③ $2x_1 + \frac{1}{2}x_2 = 12$

$(2, \frac{1}{2})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 12$

line $\perp$ to $\begin{pmatrix} 2 \\ 1/2 \end{pmatrix}$

goes through $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$

① $x_2 = 5$

$(0, 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5$

④ $x_1, x_2 \geq 0$

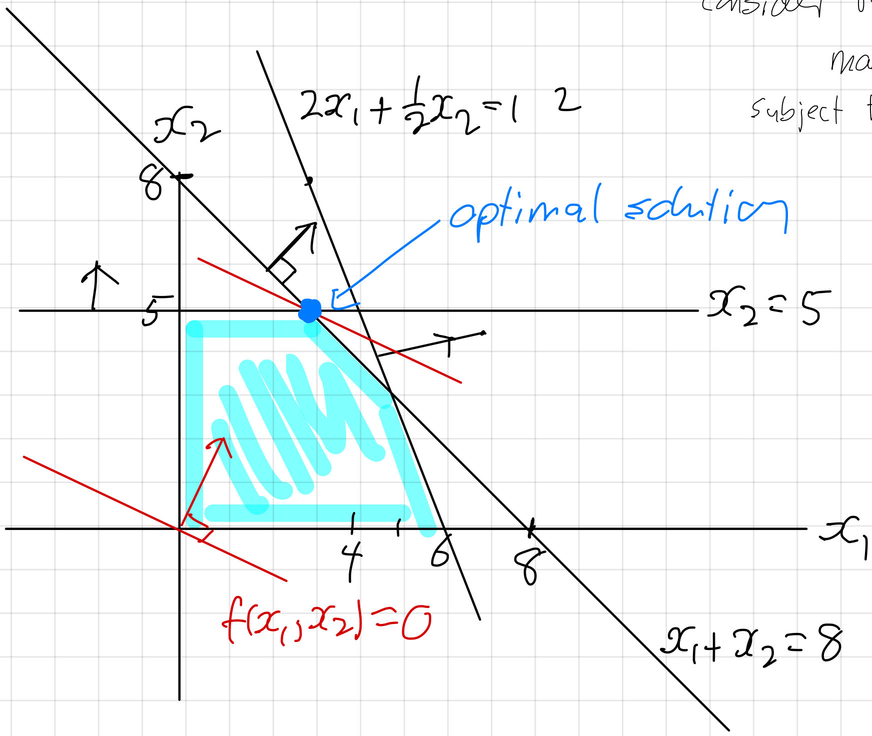
means top right quadrant

Every point in shaded region satisfies all constraints and sign restrictions

Consider following LP

maximise $x_1 + 2x_2$

subject to
$$\begin{aligned} x_2 &\leq 5 && (1)\\ x_1 + x_2 &\leq 8 && (2)\\ 2x_1 + \tfrac{1}{2}x_2 &\leq 12 && (3)\\ x_1, x_2 &\geq 0 && (4) \end{aligned}$$



Write $f(x_1, x_2) = x_1 + 2x_2$ objective function

$f(x_1, x_2) = 0$ is the line $x_1 + 2x_2 = 0$

i.e. $(1, 2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

line $\perp \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ goes through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$f(x_1, x_2) = b$ is a line parallel to $f(x_1, x_2) = 0$ and moves in the direction of the normal as b increases

Want largest b such that $f(x_1, x_2) = b$ intersects feasible region.

Any point on that line in feasible region is optimal solution

From picture $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is optimal solution.

Here objective function $= f(3, 5) = 3 + 2 \times 5 = 13$

To sketch LP in two variables

① Sketch feasible region

(i) For each constraint $a_1x_1 + a_2x_2 \leq b$
sketch the line $a_1x_1 + a_2x_2 = b$ with normal vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

If inequality of the form $a_1x_1 + a_2x_2 \geq b$
first multiply inequality by -1

(ii) The feasible region is the region bounded by those lines that is opposite to the normals and inside the quadrant given by the sign restrictions

e.g. $x_1, x_2 \geq 0$ means top right quadrant.

② Assume LP asks to maximize $c_1x_1 + c_2x_2$

Draw the line $c_1x_1 + c_2x_2 = 0$ (call it L_1)

Find any parallel line L_2 that intersects feasible region

Move L_2 in direction of normal $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ keeping it parallel until the last time it hits the feasible region. The last point(s) that it hits give the optimal solutions.

Q: how do you change ② if asked to minimise?

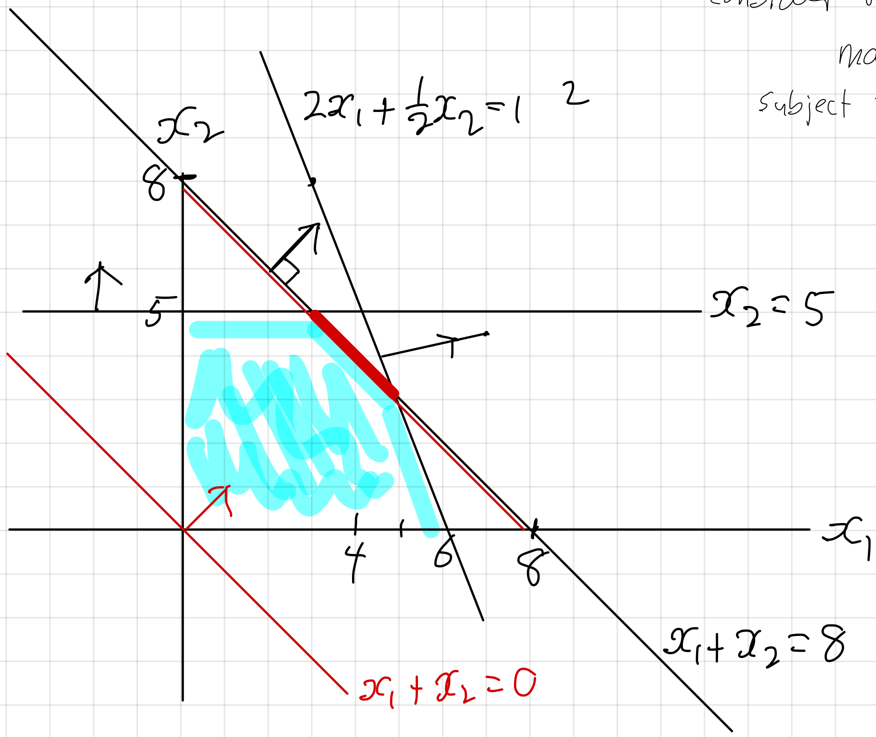
Move L_2 in opposite direction to normal

[or consider max $-c_1x_1 - c_2x_2$ ← normal $\begin{pmatrix} -c_1 \\ -c_2 \end{pmatrix}$]

Consider following LP

maximise $\cancel{x_1+2x_2}$ x_1+x_2

subject to
$x_2 \leq 5$ ①
$x_1+x_2 \leq 8$ ②
$2x_1+\frac{1}{2}x_2 \leq 12$ ③
$x_1, x_2 \geq 0$ ④



$f(x_1,x_2)=x_1+x_2$

$\perp$ (!)

Q: How would you change objective function so that there are infinitely many optimal solutions.

Change objective function so that red line is parallel to one of the constraints.

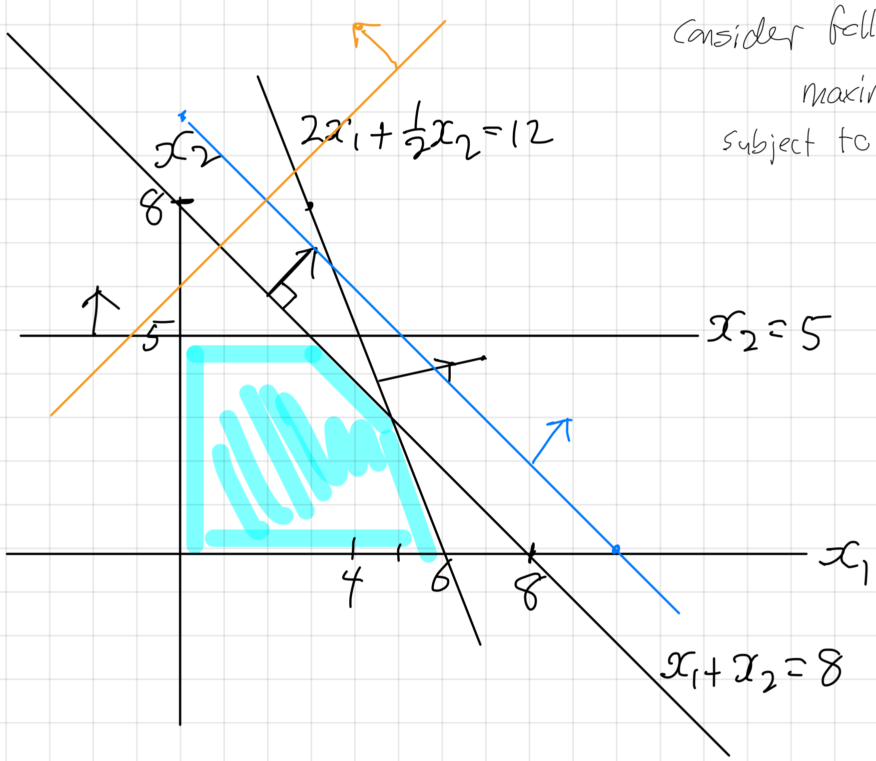
$f(x_1,x_2)=x_1+x_2$.

As we move the red line in direction of normal its final intersection with feasible region is the thick red line (so infinitely many points).

Hence infinitely many optimal solutions

Q: Can there be exactly 2 optimal solutions? No! (see Week 4).

A <u>redundant constraint</u> is one that does not affect the feasible region i.e. one we could remove from the linear program without affecting feasible region.



Consider following LP

maximise $x_1 + 2x_2$

subject to

$$x_2 \leq 5 \quad (1)$$
$$x_1 + x_2 \leq 8 \quad (2)$$
$$2x_1 + \tfrac{1}{2}x_2 \leq 12 \quad (3)$$
$$x_1, x_2 \geq 0 \quad (4)$$

Add

e.g. $x_1 + x_2 \leq 10$

$x_1 - x_2 \geq -6$ i.e. $-x_1 + x_2 \leq 6$

normal $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

From picture, adding these constraints to the picture does not change feasible region.

Q: Can an LP have zero optimal solutions?

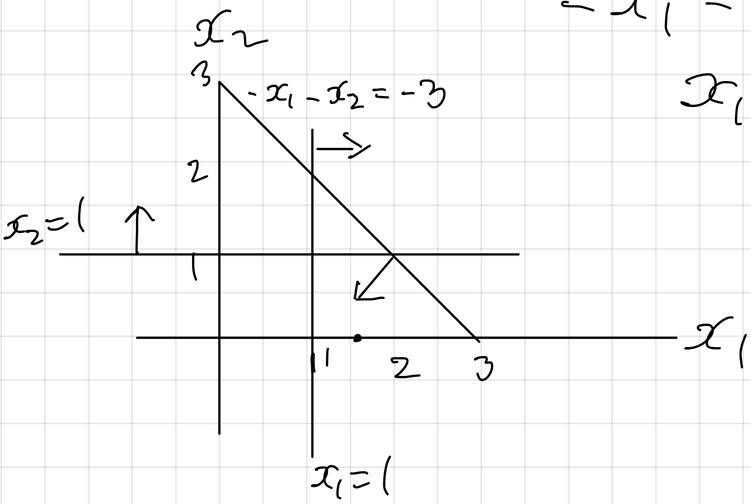
Yes. Two possibilities.

Def A linear program is called <u>infeasible</u> if it has no feasible solutions.

e.g. maximise $x_1 + x_2$

subject to

$$x_1 \leq 1 \quad (1)$$
$$x_2 \leq 1 \quad (2)$$
$$-x_1 - x_2 \leq -3 \quad (3)$$
$$x_1, x_2 \geq 0 \quad (4)$$



$-x_1 - x_2 = -3$

$(-1, -1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -3$

(1), (2) and (4) imply all feasible points are inside the unit square.

(3) says all feasible points are to the right of the diagonal line

No point in $\mathbb{R}^2$ satisfies all of these hence LP is infeasible.

Q: Change RHS of (3) so LP becomes feasible.

Replace (3) with e.g. $-x_1 - x_2 \leq 1$

Def An LP in standard inequality form

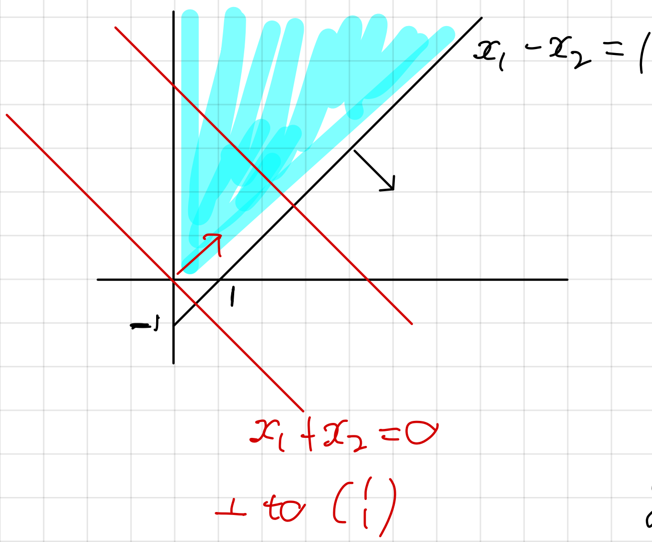
$$\text{maximise } \underline{c}^T\underline{x}$$
$$\text{subject to } A\underline{x} \leq \underline{b}$$
$$\underline{x} \geq \underline{0}$$

is called <u>unbounded</u> if for $k \in \mathbb{R}$ there is some feasible solution $\underline{c}^T\underline{x} \geq k$

e.g. maximise $x_1 + x_2$

subject to $x_1 - x_2 \leq 1$

$x_1, x_2 \geq 0$



$x_1 - x_2 = 1$

$(1, -1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1$

line $\perp$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

goes through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

objective function

The line $x_1 + x_2 = k$ (for any $k \geq 0$) intersects the feasible region so there is a feasible solution for which the objective function $x_1 + x_2$ equals k $\forall k \geq 0$. Hence this LP is unbounded. Hence no optimal solution.

How many feasible solutions are there

for which $x_1 + x_2 = 0$? 1

for which $x_1 + x_2 = 100$? ∞

for which $x_1 + x_2 = -100$? 0

Def An LP in standard inequality form

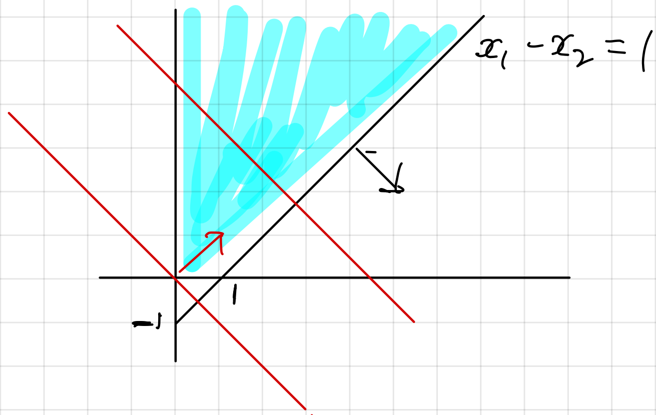
$$\begin{aligned} \text{maximise} \quad & \underline{c}^T \underline{x} \\ \text{subject to} \quad & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

is called <u>unbounded</u> if for $k \in \mathbb{R}$ there is some feasible solution $\underline{c}^T \underline{x} \geq k$

e.g. maximise $x_1 + x_2$

subject to $x_1 - x_2 \leq 1$

$x_1, x_2 \geq 0$



Q: Change the objective function so that the linear program is <u>not</u> unbounded.

e.g. either replace maximise by minimise or replace $x_1 + x_2$ with $-x_1 - x_2$

In both cases the optimal solution is $\binom{x_1}{x_2} = \binom{0}{0}$ and is unique. (This LP is <u>not</u> unbounded)

So the LP being unbounded is <u>not</u> the same as the feasible region being unbounded.

Notice in all our examples if our LP has an optimal solution, then one of the "corners" of our feasible region is an optimal solution (although there could be infinitely many others).

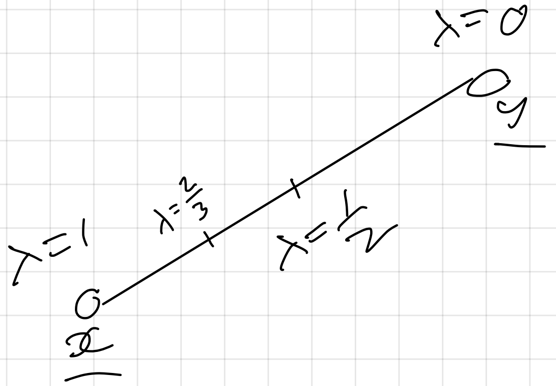
Want to formalise this.

Defn Given two vectors $\underline{x}, \underline{y} \in \mathbb{R}^n$ a convex combination of $\underline{x}$ and $\underline{y}$ is a vector of the form $\lambda \underline{x} + (1-\lambda)\underline{y}$ where $\lambda \in [0,1]$.

Notice that if $\lambda = 0$, $\lambda \underline{x} + (1-\lambda)\underline{y} = \underline{y}$

$\lambda = 1$ $\lambda \underline{x} + (1-\lambda)\underline{y} = \underline{x}$

$\lambda = \frac{1}{2}$ gives $\frac{1}{2}\underline{x} + \frac{1}{2}\underline{y}$ the point half way between $\underline{x}$ and $\underline{y}$



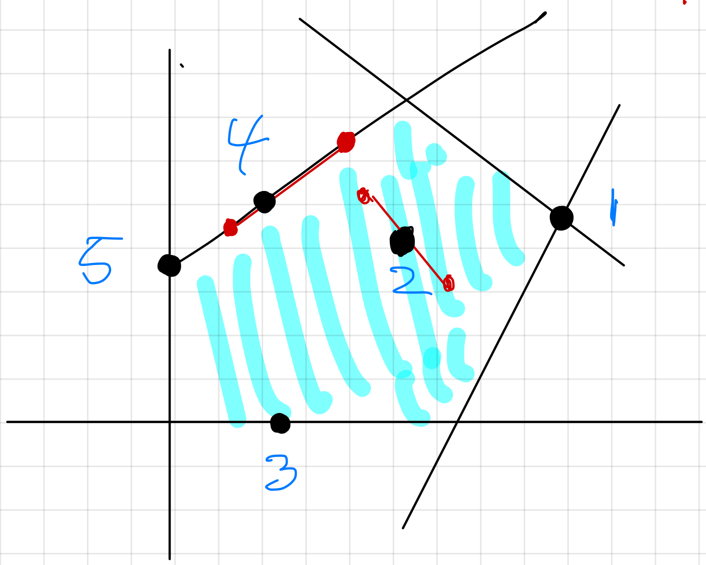
Defn Given an LP, a feasible solution $\underline{x}$ is called an <u>internal solution</u> if there exists feasible solutions $\underline{y}$ and $\underline{z}$ such that $\underline{x}$ is a convex combination of $\underline{y}$ and $\underline{z}$

(i.e. $\underline{x} = \lambda \underline{y} + (1-\lambda)\underline{z}$ for some $\lambda \in [0,1]$)

and $\underline{y} \neq \underline{x}$ $\underline{z} \neq \underline{x}$

(Note: every vector can be written as a convex combination of itself $\underline{x} = \lambda \underline{x} + (1-\lambda)\underline{x}$)

A feasible solution $\underline{x}$ is called an <u>extreme point solution</u> if it is not an internal solution (extreme point solutions are "corner" points of our feasible region).



② internal
④ internal
① extreme point
③ internal
⑤ extreme point.

This is a geometric definition of "corner".

Later we give algebraic definition.

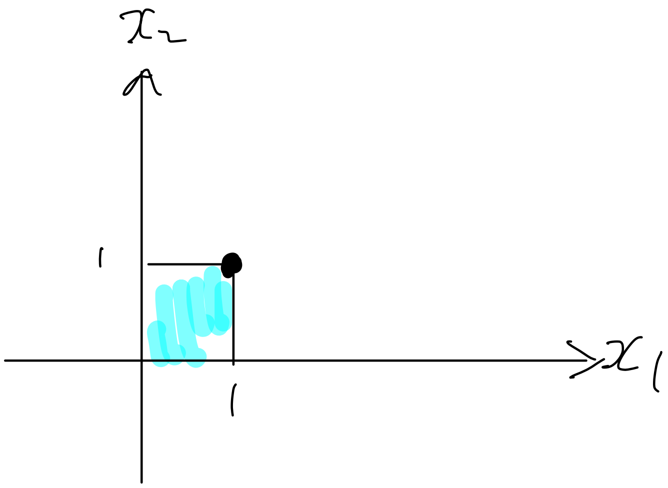
Consider LP

maximise $x_1 + x_2$

subject to $x_1 \leq 1$

$x_2 \leq 1$

$x_1, x_2 \geq 0$



Show that $\binom{1}{1}$ is an extreme point solution.

Solution (not discussed in lecture)

First $\binom{1}{1}$ is feasible since it satisfies constraints and sign restrictions.

If $\binom{1}{1}$ is internal then

$$\binom{1}{1} = \lambda \underline{a} + (1-\lambda)\underline{b}$$ for $\underline{a}, \underline{b}$ feasible where $\underline{a} \neq \binom{1}{1}$ and $\underline{b} \neq \binom{1}{1}$

Assume $\underline{a} = \binom{a_1}{a_2}$ $\underline{b} = \binom{b_1}{b_2}$

so $$\binom{1}{1} = \lambda\binom{a_1}{a_2} + (1-\lambda)\binom{b_1}{b_2} = \begin{pmatrix}\lambda a_1 + (1-\lambda)b_1 \\ \lambda a_2 + (1-\lambda)b_2\end{pmatrix}$$

We know $a_1, a_2, b_1, b_2 \leq 1$. If $a_1 < 1$ or $b_1 < 1$ then

$$\lambda a_1 + (1-\lambda)b_1 < \lambda \cdot 1 + (1-\lambda)\cdot 1 = 1$$

so $a_1 = b_1 = 1$

Similarly $a_2 = b_2 = 1$

so $\underline{a} = \underline{b} = \binom{1}{1}$ a contradiction

So extreme point soln.