Week 3-4

(continuous) GBM   (discrete) Binomial

$S(t) \leftarrow S_n(t)$ as $n \to \infty$

4. CLT & GBM

4.1 Central Limit Theorem

Let $Y_1, Y_2, \cdots, Y_n$ be a sequence of iid random variables,

$E(Y_n) = a$, $Var(Y_n) = \sigma^2$, $|Y_n| < C$.

Then the sequence $\boxed{Z_n = \dfrac{\sum_{j=1}^{n} Y_j - na}{\sqrt{n}\,\sigma}}$ converges in distribution, as $n \to \infty$ to the standard normal random variable.

Namely,

1. For all x, $\lim\limits_{n\to\infty} P(Z_n < x) = \Phi(x)$, $\underset{\text{CDF of } N(0,1)}{\Phi(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$.

2. Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying, for some $C_1 > 0$, $C_2 > 0$, the estimate $|g(x)| < C_1 e^{C_2 |x|}$.

CLT 2:

Then $\lim_{n\to\infty} E(g(Z_n)) = E(g(Z))$ where $Z \sim N(0,1)$

In other words, we have

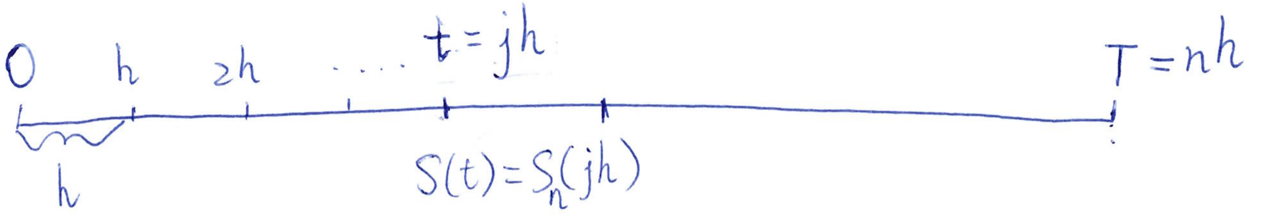
$$\underbrace{\lim_{n\to\infty} E(g(Z_n))}_{\text{discrete}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{g(x)}_{\text{continous}} e^{-\frac{x^2}{2}} dx.$$

$1 \Leftrightarrow 2$

4.2 Why does the GBM describe the behaviour of the prices?

Target: prove $\lim_{n\to\infty} \underbrace{S_n(t)}_{\text{Bin}} = \underbrace{S(t)}_{\text{GBM}}$

Step 1:



$n \to \infty$

$h \to 0$

Divide $[0, T]$ into n equal intervals of length $h = \frac{T}{n}$

① $t=0$ $\quad S_n(0) = S(0)$, where $S(0)$ is the same as in the GBM

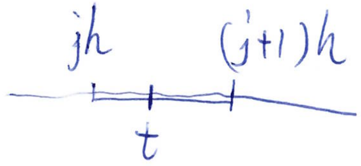
② $t = jh$, $1 \le j \le n$, set $\underline{S_n(jh)} = S(0)\, e^{\mu h j + \sigma\sqrt{h}\,(Y_1 + Y_2 + \cdots + Y_j)}$. (1)

Multiperiod Binomial model. $\qquad Y_1, \cdots, Y_j = \begin{cases} \overset{-1}{0} & d \\ 1 & u \end{cases}$

$$= S(0)\, e^{\mu h j + \sigma\sqrt{h} \sum_{k=1}^{j} Y_k}$$

$Y_1, Y_2, \cdots, Y_n, \cdots$ is a sequence of iid r.v.s $\;1$ or -1

$P(Y_j = 1) = \underset{p}{\frac{1}{2}}$, $\quad P(Y_j = -1) = \underset{1-p}{\frac{1}{2}}$



Step. 2: For $t \in [0, T]$ set $S_n(t) = S_n(jh)$ where j is s.t $jh \le t \le (j+1)h$.

Remarks.

① For each fixed n, $S_n(jh)$ evolves according to a Binomial model.

$$S_n(jh) = S_n((j-1)h)\, e^{\mu h + \sigma\sqrt{h}\, Y_j}$$

Binomial Model $\quad \boxed{S_n(jh) = \begin{cases} S_n((j-1)h)\, u \\ S_n((j-1)h)\, d \end{cases}}$

$u = e^{\mu h + \sigma\sqrt{h}} \leftarrow Y_j = 1$

$d = e^{\mu h - \sigma\sqrt{h}} \leftarrow Y_j = -1$

② Realisation of the path $S_n(jh)$, $j = 0, 1, 2, \cdots, n$

$P(\text{each path}) = 2^{-n} = \left(\frac{1}{2}\right)^n$

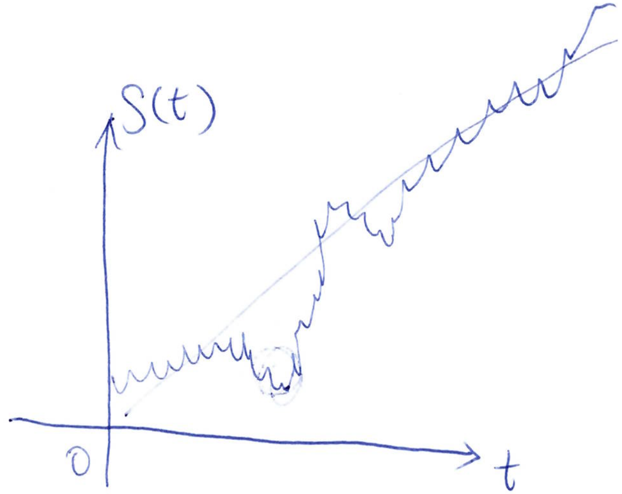
③ two key factors: $e^{\mu h}$, $e^{\pm \sigma\sqrt{h}}$

- $e^{\mu h}$: not random

  pushes price up, $\mu > 0$

- $e^{\pm\sigma\sqrt{h}}$: random

  pushes price $\left\langle \begin{array}{ll} \text{up} & \frac{1}{2} \\ \text{down} & \frac{1}{2} \end{array} \right.$



h is small, $\sqrt{h}$ much larger than h $\quad \frac{\sqrt{h}}{h} \to \infty$ as $h \to 0$

$\left\{ \begin{array}{l} \text{at each particular step: } e^{\pm\sigma\sqrt{h}} \text{ more important} \\ \text{in a long run: sum of } \pm\sigma\sqrt{h} \to 0 \quad \text{cancel out each other} \\ \qquad\qquad e^{\mu h} \text{ more important} \end{array} \right.$

Theorem 4.2 $S_n(t) \to S(t)$

When $n \to \infty$, the process $S_n(t) \to S(t)$ in the following sense:

if $g(z_1, \cdots, z_k)$ is a "good enough" function of k variables (say, continuous, growing no faster than exponentially in each variable)

and

$0 < t_1 < t_2 < \cdots < t_k \leq T$

then: $\lim\limits_{n \to \infty} E\left[g(S_n(t_1), S_n(t_2), \cdots, S_n(t_k))\right] = E\left[g(S(t_1), \cdots, S(t_k))\right]$ (4)

Proof: $k=1$ $g(S_n(t_1))$ for simplicity $z_n = S_n(t)$

Target: $E(g(S_n(t))) \xrightarrow[n \to \infty]{} E(g(S(t))) \overset{CLT2}{=\!=} \cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(S(0) e^{\mu t + \sigma\sqrt{t} x}}$

$$\overset{CLT2}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g\Big(\underbrace{S(0)\, e^{\mu t + \sigma\sqrt{t}\, x}}_{g(x) = S_n(t)}\Big)\, e^{-\frac{x^2}{2}}\, dx$$

According to the 2 Steps

$jh \le t < (j+1)h$

$$\begin{array}{ccc} jh & & (j+1)h \\ \hline & t & \end{array}$$

$t = jh \quad \because h \to 0$

$\Rightarrow h = \frac{t}{j}$

$$S_n(t) = S_n(jh) \overset{\text{Step 1}}{\overset{Eq(1)}{=}} S(0)\, e^{\mu jh + \sigma\sqrt{h}\,(Y_1 + Y_2 + \cdots + Y_j)} = \underbrace{S(0)\, e^{\mu t + \sigma \sqrt{\frac{t}{j}} \sum_{k=1}^{j} Y_k}}_{g}$$

$E(Y_j) = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0, \quad Var(Y_j) = E(Y_j^2) - [E(Y_j)]^2 = 1$

$$\sqrt{\frac{t}{j}} \sum_{k=1}^{j} Y_k = \sqrt{t} \times \boxed{\frac{1}{\sqrt{j}} \sum_{k=1}^{j} Y_k} \overset{CLT}{\longrightarrow} \sqrt{t}\, Z \quad \text{as } j \to \infty$$

$Z \sim N(0, 1)$

$a = 0,\ \sigma = 1,\ n = j$

Compare to Z_n

For a "good" function g we have

$$E\left[g\left(\frac{1}{\sqrt{j}}\sum_{k=1}^{j} Y_k\right)\right] \to E\left(g(Z)\right), \text{ where } Z \sim N(0,1)$$

In particular, as $n \to \infty$

$$E\Big(g(\underbrace{S_n(t)}_{\downarrow\; Z_n})\Big) = E\left[g\left(S(0)\cdot e^{\mu t + \sigma\sqrt{t}\frac{1}{\sqrt{j}}\sum_{k=1}^{j} Y_k}\right)\right]$$

$$\to E\left(g\left(S(0)\, e^{\mu t + \sigma\sqrt{t}\, Z}\right)\right)$$

$$\overset{\text{CLT 2}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g\left(S(0)\, e^{\mu t + \sigma\sqrt{t}\, x}\right) e^{-\frac{x^2}{2}}\, dx \qquad \blacksquare$$

Theorem 4.2 $\quad S_n(t) \longleftrightarrow S(t)$

Theorem 5.1 $\quad S(t) \longleftrightarrow \underbrace{S_n(t) \longrightarrow \tilde{S}(t)}$

Bridge $\quad$ RNP $\quad$ pricing

Suppose $S(t) = S e^{\mu t + \sigma W_t}$

r: interest rate compound continuously

Then the RNP defined on the space of function ~~$S_n(t)$~~ $S_n(t)$

converges to a RNP on the space of trajectories of

the GBM given by $\tilde{S}(t) = S e^{\tilde{\mu} t + \sigma W_t}$, $\tilde{\mu} = r - \frac{1}{2}\sigma^2$

$\downarrow$

RN GBM

$$\underbrace{S(t) \quad - \quad S_n(t)}_{\text{Th 4.2}} \quad - \quad \tilde{S}(t)$$

Th 5.1

Bridge

RNP

$\tilde{p} = u, d, r$

$\tilde{q} = 1 - \tilde{p}$

Theorem 5.2 $\boxed{C = e^{-rT} \tilde{E}}$

price RNP

Suppose that

① $R(T)$: payoff function of a derivative on a share

② T: The payoff time

③ r: interest rate compounded continuously

Then: the price $C = e^{-rT} \tilde{E}[R(T)]$

$\tilde{E}$ is the expectation over the RNP.

If we buy the derivative for C

Then Return at time T is $R(T) - Ce^{rT}$

According to the Arbitrage theorem:

$$\tilde{E}\left(R(T) - Ce^{rT}\right) = 0 \qquad (\text{Th } 2.1)$$

$$C = e^{-rT}\,\tilde{E}\left(R(T)\right) \qquad \blacksquare$$

	0	T
Borrow from the bank	C	$-Ce^{rT}$
to buy the derivative	$-C$	$R(T)$
	0	~~$-Ce^{rT} + \tilde{E}(R(T))$~~ $-Ce^{rT} + R(T)$

Theorem 5.3 $\boxed{C = e^{-rT} E(\sim)}$

Suppose that

① GBM $S(t) = Se^{\mu t + \sigma W_t}$

② Payoff function of the derivative $R\left(S(t_1), S(t_2), \cdots, S(t_K)\right)$

where $0 \le t_1 < t_2 < \cdots < t_K \le T$

③ T: payoff time

④ r: interest rate compounded continuously

Then the price C of the derivative is

$$C = e^{-rT} E\left[R(\tilde{S}(t_1), \tilde{S}(t_2), \cdots, \tilde{S}(t_K))\right],$$

$$\tilde{S}(t) = S e^{\tilde{\mu}t + \sigma W_t}, \quad \underline{\tilde{\mu} = r - \tfrac{1}{2}\sigma^2}$$

Proof: $R(T) = R\left(S(t_1), S(t_2), \cdots, S(t_K)\right)$

By Th 5.2 $\quad C = e^{-rT} \underline{\tilde{E}\left[R(S(t_1), S(t_2), \cdots, S(t_K))\right]}$

By Th 5.1 $\quad S(t) - \underline{S_n(t)} - \tilde{S}(t)$ $\quad$ Replace $S(t_1) \cdots, S(t_K)$ with $\tilde{S}(t_1), \cdots, \tilde{S}(t_K)$

$$C = e^{-rT} \underline{E\left[R(\tilde{S}(t_1), \tilde{S}(t_2), \cdots, \tilde{S}(t_K))\right]}$$

where $\tilde{S}(t) = S e^{\tilde{\mu}t + \sigma W_t}, \quad \tilde{\mu} = r - \tfrac{1}{2}\sigma^2$ ▨

5.2 Examples (applications)

5.2.1 The European Call Option

Call (K, T)

Payoff function of Call (K,T) is $R(S(T)) = (S(T)-K)^+$ ← special case

Compare to Th 5.3 : $R(S(t_1), S(t_2), \cdots, S(t_K))$ $\quad K = 1$, $t_1 = T$

$C = e^{-rT}E[\sim]$

$$\underline{C = e^{-rT} E\left[(\tilde{S}(T) - K)^+ \right]}$$

$$\tilde{S}(T) = S e^{\tilde{\mu} T + \sigma W_T}, \quad \tilde{\mu} = r - \frac{1}{2}\sigma^2$$

Calculate B-S formula $\quad \cancel{\tilde{S}(T) = \int} \quad \begin{cases} \tilde{S}(T) > K \\ \tilde{S}(T) \le K \end{cases}$

$$C = C(S, T, K, \sigma, r) = S\Phi(w) - Ke^{-rT}\Phi(w - \sigma\sqrt{T})$$

$$w = \frac{\ln\frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}, \quad \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

Remark: another form of B-S formula

$$C = e^{-rT} \int_{-\infty}^{\infty} \left(S e^{\tilde{\mu} T + \sigma \sqrt{T} x} - K \right)^{+} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$