Tutorial Week 3

Ex 4: The relativistic norm square of $\bar{A}$ is

$|\bar{A}|^2 = A^a \eta_{ab} A^b = -(A^0)^2 + 2^2$ (of the 16 terms in the sum, 12 vanish because $\eta_{ij} = 0$ if $i \neq j$ and two of the remaining 4 vanish since $\bar{A}^1 = \bar{A}^3 = 0$).

If $\bar{A}^2 = 1$ then $-(A^0)^2 + 4 = 1 \Rightarrow \bar{A}^0 = \sqrt{3}$

Then $A\,B = A^a \eta_{ab} B^b = -A^0 B^0 + \cancel{A^1 B^1} + A^2 B^2 + \cancel{A^3 B^3} =$

$= -\sqrt{3} \cdot 3 + 2\, B^2$ which vanishes if

This is the component 2 of $\bar{B}$ not the square of $\bar{B}$!

$$B^2 = \frac{3\sqrt{3}}{2}$$

Ex 5c. A spacelike vector $\bar{A}$ satisfies

$$|\bar{A}|^2 = -(A^0)^2 + \sum_{i=1}^{3} (A^i)^2 > 0 .$$

By hypothesis $\bar{A}$ and $\bar{B}$ satisfy the condition above and the goal is to understand if also $\bar{A} + \bar{B}$ satisfies it. We then have to calculate the relativistic norm of $\bar{A}+\bar{B}$ and check it it is positive assuming that $\bar{A}\bar{B}=0$. We have

$$|\bar{A}+\bar{B}|^2 = (A^a+B^a)\,\eta_{ab}\,(A^b+B^b) =$$

$$A^a\eta_{ab}A^b + A^a\eta_{ab}B^b + A^b\eta_{ab}B^b + B^a\eta_{ab}B^b =$$

$$|\bar{A}|^2 + |\bar{B}|^2 + 2\bar{A}\bar{B} = |\bar{A}|^2 + |\bar{B}|^2 > 0 \quad \checkmark$$

<u>Ex 6</u> We know that the relativistic norm is invariant under Lorentz transformations so we can calculate it in a frame F' where $B'^{1}=0$. Check that this frame exists: we can perform a boost, so we have

$$B'^{1} = -\sinh\beta\, B^0 + \cosh\beta\, B^1$$

By choosing $\tanh\beta = \dfrac{B^1}{B^0}$ we have $B'^{1}=0$.

This choice is possible since $-(B^0)^2+(B^1)^2<0$, so $-1 < B^1/B^0 < 1$. Then the

$$|\bar{A}+\bar{B}|^2 = |\bar{A}'+\bar{B}'|^2 = -(A'^0+B'^0)^2 + (A'^1)^2 =$$

$$-\underbrace{(A'^0)^2+(A'^1)^2}_{<0}-\underbrace{(B'^0)^2}_{<0}-2\underbrace{A'^0B'^0}_{}$$

since $\bar{A}$ is timelike $\quad$ obvious $\quad$ consequence of $A^0>0$ $B^0>0$

Let us check the last statement

$$A'^0=\cosh\beta\, A^0-\sinh\beta\, A^1=\cosh\beta\,(A^0-\tanh\beta\, A^1)$$

Since $|\tanh\beta|<1$ and $A^0>A^1$ we have $A'^0>0$. Of course the same holds for B'^0.

<u>Alternative derivation</u>:

$$|\bar{A}+\bar{B}|^2=-(A^0+B^0)^2+(A^1+B^1)^2=\left[-(A^0)^2+(A^1)^2\right]+$$

$$\left[-(B^0)^2+(B^1)^2\right]+2\left[-A^0B^0+A^1B^1\right]$$

The first two parenthesis are negative as $\bar{A}$ and $\bar{B}$ are timelike. This, together with $A^0, B^0>0$, implies also that $A^0>A^1$, $B^0>B^1 \Rightarrow A^0B^0>A^1B^1$, so the last square parenthosis is negative as well.

Ex 8^{bis} (A variation on 8). Consider the change of coordinates $x^0=x'^0$, $x^1=r\cos\vartheta\cos\varphi$, $x^2=r\cos\vartheta\sin\varphi$, $x^3=r\sin\vartheta$

[comment r, ϑ, φ are called polar coordinates].

Let us derive how η_{ab} changes under this change of coordinates:

$$\eta_{ab}\, dx^a dx^b = \eta_{ab} \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}\, dx'^c dx'^d$$

where $x'^1 = r$, $x'^2 = \vartheta$, $x'^3 = \varphi$. We have

Then the jacobian

row index

column index

$$\frac{\partial x^a}{\partial x'^c} = \begin{pmatrix} \frac{\partial x^0}{\partial x'^0} & \frac{\partial x^0}{\partial x'^1} & \frac{\partial x}{\partial x'^2} & \frac{\partial x^0}{\partial x'^3} \\ \frac{\partial x^1}{\partial x'^0} & \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^1}{\partial x'^3} \\ \frac{\partial x^2}{\partial x'^0} & \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^2}{\partial x'^3} \\ \frac{\partial x^3}{\partial x'^0} & \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^2} & \frac{\partial x^3}{\partial x'^3} \end{pmatrix}$$

space index

takes a diagonal form since $\frac{\partial x^0}{\partial x'^i} = 0$, $\frac{\partial x^i}{\partial x^0} =$

$$\frac{\partial x^a}{\partial x'^c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{\dfrac{\partial x^i}{\partial x'^j}} & \\ 0 & & & \end{pmatrix}$$

Focusing on the non-trivial 3x3 part, we have

$$\frac{\partial x^i}{\partial x'^j} = \begin{pmatrix} \frac{\partial x^1}{\partial r} & \frac{\partial x^1}{\partial \vartheta} & \frac{\partial x^1}{\partial \varphi} \\ \frac{\partial x^2}{\partial r} & \frac{\partial x^2}{\partial \vartheta} & \frac{\partial x^2}{\partial \varphi} \\ \frac{\partial x^3}{\partial r} & \frac{\partial x^3}{\partial \vartheta} & \frac{\partial x^3}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\vartheta\cos\varphi & -r\sin\vartheta\cos\varphi & -r\cos\vartheta\sin\varphi \\ \cos\vartheta\sin\varphi & -r\sin\vartheta\sin\varphi & r\cos\vartheta\cos\varphi \\ \sin\vartheta & r\cos\vartheta & 0 \end{pmatrix}$$

where I used $x'^1 = r$, $x'^2 = \vartheta$, $x'^3 = \varphi$ and $x^1 = r\cos\vartheta\cos\varphi$, $x^2 = r\cos\vartheta\sin\varphi$, $x^3 = r\sin\vartheta$. Thus with some patience

from $g_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\eta_{cd}$ we have

$$dx'^0 dx'^0 + \delta_{ij}\frac{\partial x^i}{\partial x'^k}\frac{\partial x^j}{\partial x'^l}dx'^k dx'^l$$

$\boxed{k=l=1}$ $\frac{\partial x^c}{\partial x'^1}\frac{\partial x^d}{\partial x'^1}\eta_{cd}$, then c, d must be space indices (since $\frac{\partial x^0}{\partial x'^1} = 0$) and must be equal (since $\eta_{ij} = \delta_{ij}$)

Then $\frac{\partial x^i}{\partial r}\frac{\partial x^i}{\partial r} = \cos^2\vartheta\cos^2\varphi + \cos^2\vartheta\sin^2\varphi + \sin^2\vartheta = 1$

$\boxed{k=1, l=2}$ Similarly in the case $k=1$, $l=2$ we have

$$\frac{\partial x^i}{\partial r}\frac{\partial x^i}{\partial \vartheta} = -r\cos\vartheta\sin\vartheta\cos^2\varphi - r\cos\vartheta\sin\vartheta\sin^2\varphi + r\cos\vartheta\sin\vartheta$$

$$= -r\cos\vartheta\sin\vartheta\left[\cos^2\varphi+\sin^2\varphi-1\right]=0$$

$\boxed{k=1,\ l=3}$ $$\frac{\partial x^i}{\partial r}\frac{\partial x^i}{\partial\varphi} = -r\cos^2\vartheta\cos\varphi\sin\varphi + r\cos^2\vartheta\sin\varphi\cos\varphi = 0$$

$\boxed{k=2,\ l=3}$ $$\frac{\partial x^i}{\partial\vartheta}\frac{\partial x^i}{\partial\varphi} = r^2\sin\vartheta\cos\vartheta\cos\varphi\sin\varphi - r^2\cos\vartheta\sin\vartheta\sin\varphi\cos\varphi =$$

$\boxed{k=2,\ l=2}$ $$\frac{\partial x^i}{\partial\vartheta}\frac{\partial x^i}{\partial\vartheta} = r^2\left[\sin^2\vartheta\cos^2\varphi+\sin^2\vartheta\sin^2\varphi+\cos^2\vartheta\right]=r^2$$

$\boxed{k=3,\ l=3}$ $$\frac{\partial x^i}{\partial\varphi}\frac{\partial x^i}{\partial\varphi} = r^2\left[\cos^2\vartheta\sin^2\varphi+\cos^2\vartheta\cos^2\varphi\right]=r^2\cos^2\vartheta$$

Thus we have

$$\eta_{ab}\,dx^a dx^b = -dx'^0 + dr^2 + r^2 d\vartheta^2 + r^2\cos^2\vartheta\, d\varphi^2$$

Thus in polar coordinate the metric is

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\cos^2\vartheta \end{pmatrix}$$

which has a different form as η_{ab} as expected, since x'^a and x^a are not related by a Lorentz transformation in this case.