Actuarial Mathematics II
MTH5125

Week 2 - Feedback: Recap with Exercises

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Feedback

I've been trying to catch up with the work I missed from the day of the assessment centres I had but there is no recording and I've been really struggling with teaching myself the content on the PowerPoints. I asked around some of my course mates and they seem equally as confused, I understand you have a timeline to teach the work in however I was wondering if you'd been able to do a brief summary of last weeks work, just going through examples as I do understand the theory itself, however we seem to find it difficult to actually apply it mathematically. Even if it's a brief summary in the first half of the lecture, I feel like we'd all benefit from it.

Net random future loss

The insurer's **net random future loss** (the present value of future loss random):

$$L_0^n = PVFB_0 - PVFP_0$$

$$L_0^n = PV \text{ of benefits outgo} - PV \text{ of premium income}$$

The net premium, generically denoted by P, may be determined according to **the principle of equivalence:**

$$\begin{aligned} E\left(L_0^n\right) &= 0 \\ E\left(PVFB_0\right) &= E\left(PVFP_0\right) \\ EPV\ of\ benefit\ outgo &= EPV\ of\ premium\ income \end{aligned}$$

$$Actuarial\ value\ of\ future\ benefits = Actuarial\ value\ of\ future\ income$$

The equivalence principle premium: whole life insurance

The net annual premium

An insurer issues a whole life insurance (annual) to a life aged x, with sum insured S (payable at the end of the year of death). Premiums are payable *annually in advance* until death. Find the net annual premium is P for a benefit $S = 100,000$ for a life $x = 60$. Consider the Standard Life Table with $i = 5\%$.

$$\begin{aligned} L_0^n &= Sv^{K_x+1} - P\ddot{a}_{\overline{K_x+1|}} = 0 \\ E\left(Sv^{K_x+1}\right) &= E\left(P\ddot{a}_{\overline{K_x+1|}}\right) \\ SA_x &= P\ddot{a}_x \end{aligned}$$

$$P = \frac{SA_x}{\ddot{a}_x}$$

$$P = 100,000\frac{0.29028}{14.9041} = 1947.652$$

The equivalence principle premium: whole life insurance

The variance of the net random future loss

$$
\begin{aligned}
L_0^n &= Sv^{K_x+1} - P\ddot{a}_{\overline{K_x+1}|} \\
&= Sv^{K_x+1} - P\frac{1-v^{K_x+1}}{d} \\
&= \left(S+\frac{P}{d}\right)\left(v^{K_x+1}\right) - \frac{P}{d}
\end{aligned}
$$

$$
\begin{aligned}
V\left(L_0^n\right) &= \left(S+\frac{P}{d}\right)^2 V\left(v^{K_x+1}\right) \\
&= \left(S+\frac{P}{d}\right)^2 \left[{}^2A_x - (A_x)^2\right]
\end{aligned}
$$

Note - in Actuarial maths 1: ${}^2A_x$ was denoted with A_x^*

The equivalence principle premium: endowment insurance

The net annual premium

$$L_0^n = Sv^{\min(K_x+1,n)} - P\ddot{a}_{\overline{\min(K_x+1,n)}|}$$

$$\begin{aligned} E\left(L_0^n\right) &= SE\left(v^{\min(K_x+1,n)}\right) - PE\left(\ddot{a}_{\overline{\min(K_x+1,n)}|}\right) \\ &= SA_{x:\overline{n}|} - P\ddot{a}_{x:\overline{n}|} \end{aligned}$$

$$\begin{aligned} E\left(L_0^n\right) &= 0 \\ SA_{x:\overline{n}|} &= P\ddot{a}_{x:\overline{n}|} \end{aligned}$$

$$P = \frac{SA_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

The equivalence principle premium: endowment insurance

The variance of net random loss

$$\begin{aligned} L_0^n &= Sv^{\min(K_x+1,n)} - P\ddot{a}_{\overline{\min(K_x+1,n)}|} \\ &= Sv^{\min(K_x+1,n)} - P\frac{1-v^{\min(K_x+1,n)}}{d} \\ &= \left(S+\frac{P}{d}\right)v^{\min(K_x+1,n)} - \frac{P}{d} \end{aligned}$$

$$\begin{aligned} V\left(L_0^n\right) &= \left(S+\frac{P}{d}\right)^2 V\left(v^{\min(K_x+1,n)}\right) \\ &= \left(S+\frac{P}{d}\right)^2 \left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right] \end{aligned}$$

Gross (expense-loaded) premiums

Investment-related expenses (e.g. analysis, cost of buying, selling, servicing)

Insurance-related expenses:

- acquisition (agents' commission, underwriting, preparing new records)
- maintenance (premium collection, policyholder correspondence)
- general (research, actuarial, accounting, taxes)
- settlement (claim investigation, legal defense, disbursement)

Gross (expense-loaded) premiums

- Most life insurance contracts incur large losses in the first year because of large **first year expenses**:
- **Renewal expenses** are expenses used for maintaining and continuing a policy
  - percentage of premium
  - per policy amount
  - combination of the two above
- **Termination expenses**: when a policy expire (death or maturity) - very small

Gross premium calculations

Treat expenses as if they are a part of benefits.
The gross random future loss at issue is

$$L_0^g = PVFB_0 + PVFE_0 - PVFP_0$$

$PVFE_0$: the present value random variable associated with future expenses incurred by the insurer.
Equivalence principle:

$$E\left(L_0^g\right) = E\left(PVFB_0\right) + E\left(PVFE_0\right) - E\left(PVFP_0\right) = 0$$

$$E\left(PVFP_0\right) = E\left(PVFB_0\right) + E\left(PVFE_0\right)$$

EPV of gross premiums $=$ EPV of benefits $+$ EPV of expenses

Exercise 6.6

A select life aged 45 purchases a fully discrete 20-year endowment insurance with sum insured \$100 000. Calculate the annual premium using the following assumptions:
i) Commission is 10% of the first premium and 2% of each subsequent premium.
ii) Other expenses are \$50 at issue and \$8 at each subsequent date.
Mortality follows Standard Select Table and $i = 5\%$.

Some helpful relations

- Reminder:

$$\begin{aligned} A_x &= 1 - d\ddot{a}_x \\ A_{x:\overline{n}|} &= 1 - d\ddot{a}_{x:\overline{n}|} \end{aligned}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}$$

$$d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i}$$

Some helpful relations

Whole life insurance:

$$A_x = A^1_{x:\overline{n}|} + v^n \; {}_np_x A_{x+n}$$
$$A_x = A^1_{x:\overline{n}|} + {}_nE_x A_{x+n}$$

Term insurance: $A^1_{x:\overline{n}|} = A_x - {}_nE_x A_{x+n}$

Some helpful relations

Endowment insurance$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + \ {}_nE_x$

Endowment insurance: $A_{x:\overline{n}|} = A_x -_n E_x A_{x+n} +_n E_x$

$$A_{[45]:\overline{20}|} = A_{[45]} -_{20} E_{[45]} A_{65} +_{20} E_{[45]}$$

Exercise 6.6

Let P be the annual premium.
When cashflows are complicated is better to value each element separately:
EPV of endowment insurance benefits:

$$\begin{aligned} 100,000A_{[45]:\overline{20}|} &= 100,000\left(A_{[45]} - {}_{20}E_{[45]}A_{65} + {}_{20}E_{[45]}\right) \\ &= 38,376.55 \end{aligned}$$

Exercise 6.6

EPV of initial and renewal expenses:

$$\begin{aligned} & 0.1P + 0.02Pa_{[45]:\overline{20}|} + 50 + 8a_{[45]:\overline{20}|} \\ = \; & 0.1P + 0.02P\left(\ddot{a}_{[45]:\overline{20}|} - 1\right) + 50 + 8\left(\ddot{a}_{[45]:\overline{20}|} - 1\right) \\ = \; & 0.08P + 42 + (0.02P + 8)\,\ddot{a}_{[45]:\overline{20}|} \\ = \; & 0.3388P + 145.53 \end{aligned}$$

Since $\ddot{a}_{[45]:\overline{20}|} = \left(1 - A_{x:\overline{n}|}\right)\frac{1.05}{0.05}$

Exercise 6.6

EPV of premiums:

$$
\begin{aligned}
P\ddot{a}_{[45]:\overline{20}|} &= P\left(1 - A_{x:\overline{n}|}\right)\frac{1.05}{0.05} \\
& 12.9409P
\end{aligned}
$$

Exercise 6.6

The equation of value/equivalent principle:

$$12.9409P = 38,376.55 + 0.3388P + 145.53$$

$$P = \$3,056.80$$

Exercise 6.1

Consider a fully discrete whole life insurance with sum insured \$200,000 issued to a select life aged 30. The premium payment term is 20 years. Assume the mortality follows the Standard Select Life Table with $i = 5\%$.

a) Write down an expression for the net loss at issue random variable

b) Calculate the annual premium.

c) Calculate the probability that the contract makes a profit

Benefits outgo: whole life insurance(annual, discrete) - \$200,000 to be paid at the end of the year of death: $v^{K_{[30]}+1}$

Income: premiums paid over 20 years - term annuity due (if not mentioned clearly in the question). If the life dies (obviously) premiums are not paid anymore.
Net future loss at issue:

$$L_0^n = 200,000 v^{K_{[30]}+1} - P\ddot{a}_{\overline{\min(K_{30}+1,20)}|}$$

Equivalence principle:

$$E\left(L_0^n\right) = 200,000 A_{[30]} - P\ddot{a}_{[30]:\overline{20}|} = 0$$

Exercise 6.1

Reminder:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - \ v^n \ {}_np_x \ddot{a}_{x+n}$$

Note that when the life reached 50 - not select anymore.

$$\ddot{a}_{[30]:\overline{20}|} = \ddot{a}_{[30]} - {}_{20}E_{[30]}\ddot{a}_{50}$$

$$P = \frac{200,000 A_{[30]}}{\ddot{a}_{[30]} - {}_{20}E_{[30]}\ddot{a}_{50}} = \frac{200,000 \times 0.07693}{13.0419} = \$1,179.74$$

Exercise 6.1

Assume the life survives more than $n > 20$ years:
Did the policy made a profit at death?
At death:

- **Benefit outgo:** 200,000
- **Income**: premiums for 20 years only - bring them to the time of death?

$$\underbrace{\ddot{a}_{[30]:\overline{20|}} \times P\,(1.05)^{20} \times}_{FV \text{ at the end of 20 years}} (1.05)^{n-20} > 200,000$$

We want the smallest integer such that the accumulations of premiums to time n exceed the sum insured

$$n = 53$$

Exercise 6.1

The probability of profit is ${}_{52}p_{30} = 0.70704$.
Note that the benefit will be paid at time 53 or layter if the life survives for 52 years from issue.