Week 3 Risk Models

$$S = X_1 + X_2 + \cdots + X_N$$

if $N=0$, $S=0$

Distribution of S

$\{\underbrace{S \le x}_{\text{CDF}} \text{ and } \underline{N=0}\}$ i.e. no claims

$\{S \le x \text{ and } N=1\}$ i.e. 1 claim of amount $\le x$

$\{S \le x \text{ and } N=2\}$ i.e. 2 claims which total $\le x$

$\vdots$

$\{S \le x \text{ and } N=r\}$ i.e. r claims which total $\le x$

$\vdots$

mutually exclusive and exhaustive

$$\{S \le x\} = \bigcup_{n=0}^{\infty} \{S \le x \text{ and } N = n\}$$

$$P\{S \le x\} = \sum_{n=0}^{\infty} P(S \le x \text{ and } N = n)$$ Traverse all N

$$= \sum_{n=0}^{\infty} P(N=n)\, P(S \le x \mid N = n\}$$

E.g. Compound distribution S:

slide6 $P(N=0) = 0.6$, $P(N=1) = 0.3$, $P(N=2) = 0.1$

X claim amount $X = \begin{cases} 1 & p = 0.5 \\ 2 & p = 0.5 \end{cases}$

$S \sim ?$

A: S: 0, 1, 2, 3, 4

$S = 0$, ~~P~~ $N = 0$

CDF

$P(S \le 0) = P(N=0) = 0.6$

$P(S \le 1) = P(S \le 0) + \underline{P(S=1)} = 0.6 + P(N=1) \cdot P(X=1 \cancel{| N=1})$

$= 0.6 + 0.3 \times 0.5 = 0.75$

$P(S \le 2) = P(S \le 1) + \underline{P(S=2)} = 0.75 + \underline{P(N=1) \cdot P(X=2)}$

$+ P(N=2) \cdot P(X_1=1) \cdot P(X_2=1)$

$= 0.75 + 0.3 \times 0.5 + 0.1 \times 0.5^2 = 0.925$

$P(S \le 3) = P(S \le 2) + P(S=3) = 0.925 + \underline{P(N=2)} \times P(X_1=1) \times P(X_2=2)$

N = 1 upper limit is 2 < 3 X

$X_1 = 1, X_2 = 2$

$X_1 = 2, X_2 = 1$

$+ P(N=2) \times P(X_1=2) \times P(X_2=1)$

$= 0.975$

$P(S \le 4) = P(S \le 3) + P(S=4) = 0.975 + P(N=2) \cdot P(X_1=2) \cdot P(X_2=2)$

$= 1$

W3 3

Moments of Compound distributions

$E(S)$, $Var(S)$

The law of total expectation

$$E(S) = E[E(S|N)]$$

$$E(S) = \int_0^\infty s\, f(s)\, ds$$

$$E(S|N) = \int_0^\infty s\, f(s|N)\, ds$$

Proof: $RHS = E[\underbrace{E(S|N)}] = E\left[\int_0^\infty s\, f(s|N)\, ds\right]$

N discrete

$$= \sum_{\text{all } n} P(N=n) \int_0^\infty s\, f(s|N=n)\, ds$$

$$= \int_0^\infty s \underbrace{\sum_{\text{all } n} P(N=n)\, f(s|N=n)}_{f(s)}\, ds$$

$$= \int_0^\infty s\, f(s)\, ds = E(S) = LHS$$

The law of total variance

$$Var(S) = E[Var(S|N)] + Var[E(S|N)]$$

Proof: $LHS = Var(S) = E(S^2) - [E(S)]^2 = E[E(S^2|N)] - [E[E(S|N)]]^2$

use the law of total expectation

$$= E[Var(S|N) + (E(S|N))^2] - [E[E(S|N)]]^2$$

$Var(S) = E(S^2) - [E(S)]^2$

$Var(S|N) \overset{②}{=} E(S^2|N) - [E(S|N)]^2$

$$= E[Var(S|N)] + E[E(S|N)]^2 - [E[E(S|N)]]^2$$

$$= E[Var(S|N)] + Var(E(S|N))$$ ①&②

$E[E(S|N)]^2 \overset{①}{=} E[E(S^2|N)]$

Linearity of E func

$$= RHS$$

$\cancel{E[[E(S|N)]^2]}$

$\cancel{= E[E(S|N)]^2}$

$\cancel{E[[E(S)]^2] = E[E(S)]^2}$

The mean $E(S)$

$E(S) = E(E(S|N))$

$E(S|N=n) = \sum_{i=1}^{n} E(X_i) = nm_1$

fix $n \leftarrow$ constant

N: r.v.

$E(S|N) = Nm_1 \ (*)$

$$\begin{cases} m_1 = E(X_i) \\ m_2 = E(X_i^2) \end{cases}$$

$E(S) = E[E(S|N)] = E[Nm_1]$

$= E(N)\cdot m_1$

$$\boxed{E(S) = E(N)\cdot E(X)}$$

Q: $X \sim$ Pareto (α, λ), $\alpha = 3$, $\lambda = 400$

$S = \sum_{i=1}^{N} X_i$

$N \sim$ Poisson (50)

$E(X) = \dfrac{\lambda}{\alpha - 1}$

Find $E(S)$

A: $E(S) = E(N)\cdot E(X) = 50 \times \dfrac{400}{3-1} = 10{,}000$

The variance $Var(S)$

$$Var(S|N=n) = Var\left[\sum_{i=1}^{n} X_i\right] \underset{\uparrow \text{ independence}}{=} \sum_{i=1}^{n} Var(X_i) \underset{\uparrow \text{ identity}}{=} n(\overset{E(X^2)}{m_2} - \overset{E(X)}{m_1^2})$$

fix n

$$\underline{Var(S|N)} = \underline{N(m_2 - m_1^2)} \quad (**)$$

$$Var(S) = E(\underline{Var(S|N)}) + Var[E(S|N)]$$

$$= E[\overset{(**)}{N(m_2 - m_1^2)}] + Var\overset{(*)}{(Nm_1)}$$

$$= E(N)(m_2 - m_1^2) + Var(N)m_1^2$$

$$\boxed{Var(S) = E(N)\,Var(X) + Var(N)\,[E(X)]^2}$$

MGF of S

$$M_S(t) = E(e^{tS})$$

$$M_S(t) = E\left[\underbrace{E(e^{tS} \mid N)}\right] \leftarrow E(X) = E\left[E(X \mid N)\right]$$

$$X \leftarrow e^{tS}$$

$$E(e^{tS} \mid N = n) = E\left[e^{tX_1 + tX_2 + \cdots + tX_n}\right]$$

$\{X_i\}_{i=1}^{n}$ independent identical

$$\overset{\text{independence}}{=} \prod_{i=1}^{n} E(e^{tX_i})$$

$$\overset{\text{identity}}{=} \prod_{i=1}^{n} M_X(t) = [M_X(t)]^{n}$$

$$\underbrace{E(e^{tS} \mid N)} = [M_X(t)]^{N} \leftarrow$$

N: r.v.

func of N

$$M_S(t) = E\left[E(e^{tS} \mid N)\right] = E\left[[M_X(t)]^{N}\right]$$

$$= E\left[\exp\{\ln [M_X(t)]^{N}\}\right] = E\left[\exp\{N \underbrace{\ln M_X(t)}\}\right]$$

$$= M_N(\ln M_X(t))$$

Def of $M_N(t)$

$$\downarrow$$

$$E(e^{Nt}) \triangleq M_N(t)$$

$$\boxed{M_S(t) = M_N(\ln M_X(t))}$$

Q: Slide 18

A: a)

$$\underline{E(S) = E(N) \cdot E(X)}$$

$$E(S) = E(N) \cdot B$$

$$Var(S) = \cancel{E[Var(S|N)] + Var[E(S|N)]}$$

$$\underline{Var(S) = E(N)\, Var(X) + Var(N)[E(X)]^2}$$

$$Var(S) = E(N) \times 0 + Var(N) \cdot B^2$$

$$= Var(N) \cdot \beta^2$$

b) S: $\underset{N=0}{0}$, $\underset{1}{B}$, $\underset{2}{2B}$, ..., $S = BN$

$$P(S \le s) = P(BN \le s) = P\left(N \le \frac{s}{B}\right)$$

The Compound Poisson distribution

$N \sim \text{Poisson}(\lambda)$

Facts of N: $E(N) = Var(N) = \lambda$

$$M_N(t) = e^{\lambda(e^t - 1)}$$

$$\boxed{E(S)} = E(N) \cdot E(X) = \boxed{\lambda \cdot m_1}$$

$$m_1 = E(X)$$

$$m_2 = E(X^2)$$

$$\boxed{Var(S)} = E(N) \cdot Var(X) + Var(N) \cdot [E(X)]^2$$

$$= \lambda (m_2 - m_1^2) + \lambda m_1^2$$

$$\boxed{= \lambda m_2}$$

$m_2 \neq Var(X)$

$$\boxed{M_S(t)} \overset{\text{formulae}}{=} M_N(\ln M_X(t))$$

$$M_N(t) \overset{def}{=} e^{\lambda(e^t - 1)}$$

$\ln M_X(t)$

$$= e^{\lambda(e^{\ln M_X(t)} - 1)}$$

$$\boxed{= e^{\lambda(M_X(t) - 1)}}$$

$$E(N) = Var(N) = \lambda$$

$$\boxed{skew(S)} = E([S - E(S)]^3) = \boxed{\lambda m_3}$$

Sums of independent Compound Poisson r.v.s

$A = S_1 + S_2 + \cdots + S_n$ $\quad S_i \sim$ Compound Poisson (λ_i)

CDF: $F_i(x)$

① $A \sim$ compound Poisson (Λ)

$$\Lambda = \sum_{i=1}^{n} \lambda_i$$

② $$F(x) = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i F_i(x)$$

The Compound Binomial distribution $\quad N \sim Bin(n, p)$

Facts of (Bin) N: $E(N) = np$

$Var(N) = np(1-p)$

$M_N(t) = (pe^t + 1 - p)^n$

	mean	Var
Normal	μ	σ^2
Poisson	λ	λ

$$E(S) = E(N) \cdot E(X) = npm_1$$

$$Var(S) = E(N)\,Var(X) + Var(N)\left(E(X)\right)^2 = np\left(m_2 - m_1^2\right) + np(1-p)m_1^2$$

$$= npm_2 - np^2 m_1^2$$

$$M_S(t) \overset{\text{formulae}}{=} M_N\left(\underbrace{\ln M_X(t)}\right) \qquad M_N(t) = \left(pe^{t} + 1 - p\right)^n$$

$$= \left(pe^{\ln M_X(t)} + 1 - p\right)^n$$

$$= \left(pM_X(t) + 1 - p\right)^n$$

Skew (S) Slide. 26

Example Slide. 27

A: (a) $M_X(t) = \left(1 - \frac{t}{0.2}\right)^{-10}$, $M_N(t) = \left(0.99 + 0.01e^{t}\right)^{100}$

$$\boxed{M_S(t) = M_N\left(\ln M_X(t)\right)} = \left(0.99 + 0.01e^{\ln\left(1-\frac{t}{0.2}\right)^{-10}}\right)^{100} = \cdots$$

Slide 28

Gamma: mean $= \frac{\alpha}{\beta}$, Var $= \frac{\alpha}{\beta^2}$

(b) $E(S) = E(N) \cdot E(X) = np \cdot m_1 = 100 \times 0.01 \times \frac{10}{0.2} = 50$

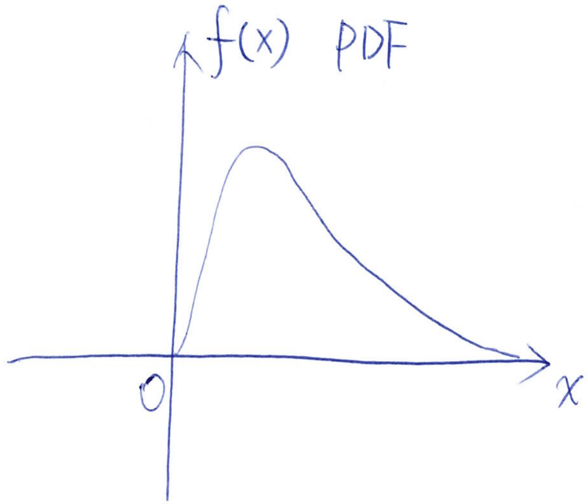
$Var(S) = E(N) \cdot Var(X) + Var(N) \cdot (E(X))^2 = np(m_2^2 - m_1^2) + np(1-p) m_1^2$

$= np m_2 - np^2 m_1^2$

$m_2 = E(X^2) = Var(X) + (E(X))^2 = \underset{\substack{\uparrow \\ Var}}{\frac{\alpha}{\beta^2}} + \underset{\substack{\uparrow \\ E}}{\left(\frac{\alpha}{\beta}\right)^2} = \frac{\alpha(\alpha+1)}{\beta^2}$

$Var(S) = 100 \times 0.01 \times \frac{10 \times (10+1)}{0.2^2} - 100 \times 0.01^2 \times \left(\frac{10}{0.2}\right)^2 = 2725$

X:



Positively-skewed