WEEK 3

§4 Taylors Theorem

$$f^{(n+1)}(x) = \frac{d}{dx}\left(f^{(n)}(x)\right)$$

§4.1 Higher Order Derivative

If the function $f': \Omega \to \mathbb{R}$ is again differential, we can consider $f'': \Omega \to \mathbb{R}$ $\frac{d^2f}{dx^2} = \frac{d}{dx}(f'(x))$

Definition 4.1.1 Let $f: \Omega \to \mathbb{R}$ be n-times differential at $x_0 \in \Omega$ if for some $n \in \mathbb{N}$, the n-th derivative exists.

$\frac{df}{dx}(x) = f'(x)$
$= f^{(1)}(x)$

We call $f^{(k)}$ the k-th derivative is given by

$$f^{(0)}(x_0) = f(x_0), \quad f^{(k+1)}(x_0) = f^{(k)'}(x_0)$$

We say f is n-times continuously differentiable at x_0 if $f^{(n)}(x_0)$ is continuous at x_0

For n "small", we will often write $f^{(2)} = f''$

§4.2 Taylor Series

Def^n 4.2.1 Let f be defined on some interval containing 0.

If f is k-times diff at $x=0$, then the sum

$$\sum_{n=0}^{k} \frac{f^{(n)}(0)x^n}{n!}$$

is called the **Taylor sum** to **order k** at **$x=0$**.

if f is infinitely differentiable at 0 (derivatives of all orders)

then the series $\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$ is called the Taylor series of f

The remainder $R(f,x) = f(x) - \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$

Note $f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$ iff $R(f,x) = 0$.

§4.2.2. Power series expansions

Suppose that $f(x)$ can be written as a power series, i.e.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Note:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow f(0) = a_0$$

$$f^{(1)}(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow f^{(1)}(0) = 1.a_1$$

$$f^{(2)}(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \Rightarrow f^{(2)}(0) = 2.1.a_2$$

Inductive argument gives $f^{(n)}(0) = n! a_n$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Example 1 $f(x) = \sin(x) = \sum a_n x^n$

$f^{(0)}(0) = \sin(0) = 0$
$f^{(1)}(0) = \cos(0) = 1$
$f^{(2)}(0) = -\sin(0) = 0$
$f^{(3)}(0) = -\cos(0) = -1$
$f^{4}(0) = \sin(0) = 0$

DERIVATIVES REPEAT EVERY 4th differentiation!

$f^{(5)}(0) = f^{(1)}(0)$, etc.
$f^{(6)}(0) = f^{(2)}(0)$, etc

So $f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$

Remainder $= 0$

✓ correct!

See later

Example 2

$f(x) = e^x$, $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$

Remainder = 0

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

✓ correct !

See later !

Example 3

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad = \exp\left(-1/x^2\right)$$

$g(y) = \exp(y)$; $y = h(x) = \frac{-1}{x^2}$.

$$f'(x) = e^{-1/x^2}\left(\frac{2}{x^3}\right) \qquad f^{(2)}(x) = e^{-\frac{1}{x^2}}\left(\frac{-6}{x^4}\right) + e^{-\frac{1}{x^2}}\left(\frac{2}{x^3}\right)^2$$

typical terms are $\frac{e^{-1/x^2}}{x^k} = \frac{1}{e^{1/x^2} \cdot x^k}$ for some $k > 0$

Observe

$$0 \leq \frac{1}{|e^{1/x^2} \cdot x^k|} = \left|\frac{1}{\left(1 + \frac{1}{x^2} + \frac{1}{2!}\left(\frac{1}{x^2}\right)^2 + \frac{1}{3!}\left(\frac{1}{x^2}\right)^3 + \cdots\right)} \cdot \frac{1}{x^k}\right| \qquad x \neq 0$$

$$\leq \frac{1}{\frac{1}{n!}\left(\frac{1}{x^2}\right)^n} \cdot \frac{1}{x^k} = \frac{n!\,|x|^{2n}}{|x|^k} = n!\,|x|^{2n-k}$$

So $\frac{1}{e^{1/x^2} x^k} \to 0$ as $x \to 0$ since

$n! \, |x|^{2n-k} \to 0$ as $x \to 0$, provided we choose $2n > k$.

$\therefore f^{(n)}(x) \to 0$ as $x \to 0$

$f(0) = 0$

$$f^{(1)}(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0.$$

$$f^{(2)}(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \to 0} \frac{f'(x)}{x} = 0$$

$\vdots$

$f^{(n)}(0) = 0$

$e^{-1/x^2} = 0 + 0 + 0 + \dots = 0$!!

$$\text{Remainder} = f(x) - \sum_{n=0}^{\infty} \frac{f^{(n)}(0) x^n}{n!} = 0$$

X

See later.

Theorem 4.2.2 (Taylor's Theorem)

Assume that f is $(k+1)$-times differentiable on some open interval containing $[0,a]$ and let $x \in (0,a)$. Then for some $c \in (0,x)$, we have

$$f(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)x^2}{2!} + \cdots + \frac{f^{(k)}(0)x^k}{3!} + R_{(k,a)}f(x)$$

Such that

$$R_{(k,0)}f(x) = \frac{f^{(k+1)}(c)\,x^{k+1}}{(k+1)!}$$

Generalised MVThm to order $(k+1)$

Proof Fix $x \neq 0$, let M solve the equation

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n + \frac{M x^{k+1}}{(k+1)!}$$

for $x \in (0, a)$.

We need to show that $M = f^{k+1}(c)$, for some $c \in (0, x)$.

Define

$$g(t) = \sum_{n=0}^{k} \frac{f^{(n)}(0) t^n}{n!} + \frac{M t^{k+1}}{(k+1)!} - f(t)$$

where $t \in (0, x)$.

By direct substitution $g(0) = g(x) = 0$

$\therefore$ by MVT $\exists$ x_1 such that

$$g^{(1)}(x_1) = g'(x_1) = 0 \quad \text{with } 0 < x_1 < x < a$$

Now consider

$$g^{(1)}(t) = \sum_{n=1}^{k} \frac{f^{n}(0)}{n!} n t^{n-1} + \frac{Mt^k}{k!} - f'(t)$$

$$g^{(1)}(0) = \sum_{n=1}^{k} \frac{f^{(n)}(0)}{n!} n(0)^{n-1} + 0 - f'(0)$$

$$= f^{(1)}(0) - f^{(1)}(0) = 0.$$

and $g^{(1)}(x_1) = 0$

$\therefore \exists x_2 \in (0, x_1)$ such that $g^{(2)}(x_2) = 0$

(Repeating this differentiation reduces $\frac{Mt^{k+1}}{(k+1)!} \to \frac{Mt^k}{k!} \to \frac{Mt^{k-1}}{(k-1)!} \to \cdots \to \frac{Mt}{1!}$ and finally to M, a constant after $(k+1)$ steps ⊛)

Hence we get repeatedly for $m < k+1$

$$g^{(m)}(x_m) = 0 \quad \text{for } m < k+1 \text{ for a sequence}$$

of values x_m satisfying

$$0 < x_{k+1} < x_k < x_{k-1} \cdots < x_1 < x$$

by repeated use of Rollé's theorem.

Now $g^{(k+1)}(x_{k+1}) = 0$, and

But $g^{(k+1)}(t) = M - f^{(k+1)}(t)$ (see ⊛)

$$\Rightarrow f^{(k+1)}(x_{k+1}) = M$$

Let $c = x_{k+1}$ & we get $\boxed{M = f^{(k+1)}(c)}$, $c \in (0, x)$

∎

Notation For the general expansion about $x = a$

$$T_{(k,a)} f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

The remainder satisfies

$$R_{(k,a)} f(x) \left(= R_{(k,a)}(x)\right) = f(x) - T_{(k,a)} f(x)$$

&

$$R_{(k,a)} f(x) = R_{(k,a)}(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$$

This is the LAGRANGE form of the remainder

Making $f(x)$ the subject we have

$$f(x) = T_{(k,a)} f(x) + R_{(k,a)} f(x)$$

4.2.3 Examples

(i) $f(x)=e^x$, $f^{(1)}(x)=e^x$, $f^{(n)}(x)=e^x$, $e^0=1$.

$\therefore$ $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ if $\lim_{k\to\infty} R_{k,0}(x)=0$

Lagrange Remainder is

$f^{(0)}(x)=e^x$

$f^{(k+1)}(x)=e^x$

$$|R_{(k,0)}f(x)| = \frac{f^{(k+1)}(c)\,x^{k+1}}{(k+1)!}$$

$$= \frac{e^c x^{k+1}}{(k+1)!},$$ for some $c\in(0,x)$.

Now $0 < c < x$

$$|R_{(k,0)}f(x)| < \left|\frac{e^x \cdot x^{k+1}}{(k+1)!}\right| \overset{?}{\to} 0 \quad \text{as } k \to \infty.$$

Let $K > |x|$

$$|R_{(k,0)}f(x)| < \frac{e^x |x|^K}{K!} \cdot \frac{|x|}{K+1} \cdot \frac{|x|}{K+2} \cdots \frac{|x|}{k} \cdot \frac{|x|}{k+1}$$

"fixed"

↓

$$< \frac{e^x |x|^K}{K!} \left(\frac{|x|}{K+1}\right)^{k-K}$$

constant

constant < 1

→ larger powers as $k \to \infty$

→ 0 as $k \to \infty$

Finally $\left(\frac{|x|}{K+1}\right)^k \to 0$ as $k \to \infty$ since $\frac{|x|}{K+1} < 1$

$$\therefore \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Conclusion!

(ii) $f(x) = \sinh(x)$ (note $\sinh'(x) = \cosh(x)$ & $\cosh'(x) = \sinh(x)$)

$f^{(2n)}(x) = \sinh(x)$

$f^{(2n+1)}(x) = \cosh(x)$

$n \in \mathbb{N}$

$$f(x) = \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Assume: $x > 0$, Lagrange Remainder is:

$$|R_{(k,0)} f(x)| = \left| \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1} \right|$$

Note

$$\sinh(x) = \frac{e^x - e^{-x}}{2} < \frac{2e^x}{2} = e^x$$

"or $\cosh(c)$"

$$= \left| \frac{\sinh(c)}{(k+1)!} x^{k+1} \right| < \left| \frac{e^c x^{k+1}}{(k+1)!} \right| < \left| \frac{e^x x^{k+1}}{(k+1)!} \right|, \quad 0 < c < x$$

Example $f(x) = \sin x$

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = \cos x$

$$T_{(k,0)} f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n + R_{(k,0)} f(x)$$

$$R_{(k,0)} f(x) = f^{(k+1)}(c) \frac{x^{k+1}}{(k+1)!}$$

$f(x) = \sin x$, $\therefore$ $|f^{(n)}(x)| = \begin{cases} |\sin(x)| & n \text{ even} \\ |\cos(x)| & n \text{ odd.} \end{cases}$

$\because$ $|f^{(n)}(x)| \leq 1, \forall x \Rightarrow |R_{(k,0)} f(x)|$ $\leq 1 \frac{|x|^{k+1}}{(k+1)!}$

use $|\sin x| \leq 1$
$|\cos x| \leq 1$
for all x

$\to 0$ as $k \to \infty$

cf. example of $f(x) = e^x$

Example $f(x) = \ln(1+x)$

$$f^{(0)}(0) = 0, \quad f^{(1)}(x) = \frac{1}{1+x}, \quad f^{(2)}(x) = \frac{-1}{(1+x)^2},$$

$$f^{(3)}(x) = \frac{2}{(1+x)^3}, \; \cdots\cdots, \quad f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}$$

$\therefore$

$$T_{(k,0)}f(x) = 0 + \frac{1}{1!}\cdot x - \frac{1!\,x^2}{2!} + \frac{2!\,x^3}{3!} - \frac{3!\,x^4}{4!} + \cdots + (-1)^{k+1}\frac{x^k}{k}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + (-1)^{k+1}\cdot\frac{x^k}{k}$$

$$\left|R_{(k,0)}f(x)\right| = \left|\frac{x^{k+1}}{(k+1)!}\cdot f^{(k+1)}(c)\right|, \qquad 0 < c < x.$$

where,

$$\left|f^{(k+1)}(c)\right| = \left|\frac{k!}{(1+c)^k}\right|$$

$\frac{1}{(1+c)^k} < 1$ for $c > 0$

$\therefore \quad |R_{(k,0)} f(x)| = \dfrac{|x|^{k+1}}{|1+c|^{k}} \dfrac{1}{(k+1)} < \dfrac{|x|^{k+1}}{(k+1)}$

If we assume $|x| < 1$, then $\dfrac{|x|^{k+1}}{(k+1)} \to 0$ as $k \to \infty$

$\therefore \quad \ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$

for $|x| < 1$.

Example

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0 & x \leq 0 \end{cases}$$

k-th Lagrange remainder $\not\to 0$ as $k \to \infty$. Why?

Show $f^{(n)}(0) = 0 \quad \forall n$. $T_{(\infty,0)}f(x) \equiv 0$, $f(x) \not\equiv 0$

$\therefore R_{(\infty,0)}f(x) \not\equiv 0$

NOTE

$$f^{(1)}(x) = e^{-1/x}\frac{1}{x^2}, \quad f^{(2)} = e^{-\frac{1}{x}}\left(\frac{1}{x^2}\right)^2 + e^{-1/x}\left(\frac{-2}{x^3}\right)$$

$$f^{(k)}(x) = e^{-\frac{1}{x}}\left(\text{polynomial in } \left(\frac{1}{x}\right)\right)$$

You can prove $\frac{f^{(k)}(x)}{x} \to 0$

- hence $f^{(k+1)}(0) = 0 \Rightarrow T_{(n,0)}f(x) = 0$!

END OF WEEK 3