Week 3 Monday lecture

PLAN: (1) Spacelike, timelike and null vectors

(2) Proper time

(3) Tensors

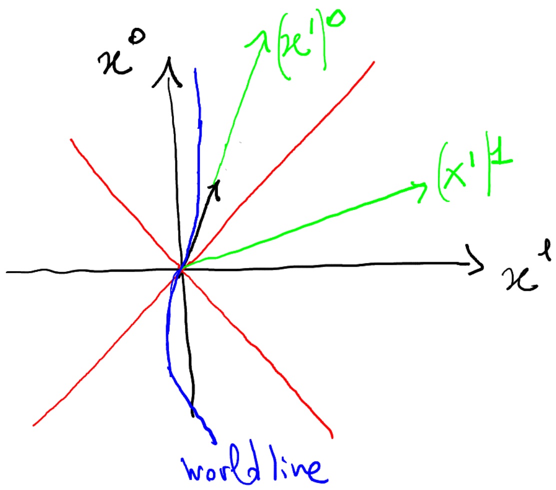
(1) We know that Lorentz transformations do not change the relativistic length^2 of vectors and in general the scalar product among spacetime vectors ($v^a \eta_{ab} w^a = (v')^a \eta_{ab} (w')^b$ where v', w' are related to v, w by a Lorentz transformation).

Thus we can divide vectors in 3 categories

- timelike if they satisfy $v^a \eta_{ab} v^b < 0$
- spacelike if they satisfy $v^a \eta_{ab} v^b > 0$
- null if they satisfy $v^a \eta_{ab} v^b = 0$

Notice that nature (space/time/null-like) of a vector does not change under a Lorentz transform., so all inertial observers agree on this classification.

As an example consider a particle whose worldline in spacetime is in the plane (x^0, x^1)



We can parametrize the world line with a parameter τ, i.e. the blue curve is described by $x^a(\tau)$

In the picture the black vector tangent to the world line at $x^0(\tau_0) = x^1(\tau_0) = 0$ is timelike.
This vector is $\left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}\right)_{\tau=\tau_0}$ and since its time component $\left.\frac{dx^0}{d\tau}\right|_{\tau_0}$ is larger than the space one, we have $\left(\frac{dx^0}{d\tau}\right)^2_{\tau_0} \geq \left(\frac{dx^1}{d\tau}\right)^2_{\tau_0}$ which implies

$$\left[\left(\frac{dx^a}{d\tau}\right)\eta_{ab}\left(\frac{dx^b}{d\tau}\right)\right]_{\tau_0} = -\left(\frac{dx^0}{d\tau}\right)^2_{\tau_0} + \left(\frac{dx^1}{d\tau}\right)^2_{\tau_0} < 0$$

We know the red lines depict the light-cone and all null vectors at $(0,0)$ lie along this "surface".

Exercises a) Depict a timelike and a null-line vector at $(0,0)$ in the figure above

b) Show that the space/time/null-line nature of a vector is invariant undere reparametrisation of the worldline, i.e. you can choose a parameter τ' which is a bijective function of τ such that $\frac{d\tau'}{d\tau} \neq 0$ and $\frac{dx^a}{d\tau'}$ is still time live.

c) The time-like vector in the figure is <u>future-directed</u> since its time component $\frac{dx^0}{d\tau}$ is positive (analogously time-like vectors with a negative time component are said to be <u>past-directed</u>). Find a Lorentz transformation that swap future and past directed time-live vectors.

(d) The velocity is $\frac{\left(\frac{dx^1}{d\tau}\right)}{\left(\frac{dt}{d\tau}\right)}$. Does it depend on the parametrisation?

(2) Consider an inertial observer who sees the particle at rest at $(0,0)$. The inertial frame for this observer is depicted in green in the figure above and the particle has zero velocity at $t'=t=0$. We indicate as $dx^a(\tau)$ an infinitesimal change of the worldline. At $t=t'=0$ we have

Green frame F' $\quad dx'^a(\tau) = (dx'^0, 0)$

Black " $\quad F$ $\quad dx^a(\tau) = (dx^0, dx^1)$ $\quad \Rightarrow$

$$-(dx'^0)^2 = -(dx^0)^2 + (dx^1)^2$$

It is convenient to define the "proper time" $d\tau$ $c\,d\tau = dx'^0$. In general for time-like worldline we have

$$\underbrace{-(dx'^0)^2}_{-(c^2 d\tau^2) \rightsquigarrow \text{ proper time}^2} = -(dx^0)^2 + \sum_{i=1}^{3} (dx^i)^2 \equiv -\underset{\substack{\uparrow \\ \text{invariant} \\ \text{length}^2}}{ds^2}$$

We can take the same approach at any point along the worldline and parametrise it with the proper time. Suppose you start from a random parametrisation $x^a(\hat{\tau})$. Then $x^a(\tau)$ satisfies

$$\frac{dx^a}{d\tau} \eta_{ab} \frac{dx^b}{d\tau} = -c^2 = \frac{dx^a}{d\hat{\tau}} \eta_{ab} \frac{dx^b}{d\hat{\tau}} \left(\frac{d\hat{\tau}}{d\tau}\right)^2 \Rightarrow$$

$$\frac{d\tau}{d\hat{\tau}} = \frac{1}{c}\left(-\frac{dx^a}{d\hat{\tau}} \eta_{ab} \frac{dx^b}{d\hat{\tau}}\right)^{1/2} = \frac{ds}{d(c\hat{\tau})}$$

Solving this differential equation yield $\tau(\hat{\tau})$, i.e.

the relation between the initial parametrization and the one with the proper time

(3) So far we saw objects with one upper spacetime index (the coordinates x^a of a spacetime event), but also objects with two lower indices such as the Minkowski metric η_{ab} or its inverse η^{ab} with two upper indices. The η's can be used to lower/raise indices and relate x^a to x_a.

It is now time to think about some general properties of objects with several upper and lower indices $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell}$. Let us start from the known case of the vector coordinates x^a. It is convenient to introduce the combination $X = x^a \frac{\partial}{\partial x^a}$ (where, as usual, we are summing over the repeated indices). We call X a vector whose coordinates are x^a. A nice bonus of this definition is that X acts naturally on functions $\phi(x)$ and calculates the derivative of ϕ's along the direction defined by the vector

$$\left(x^a \frac{\partial}{\partial x^a}\right) \phi = x^a \frac{\partial \phi}{\partial x^a}$$

Another advantage is that it encodes how the x^a's behave under change of coordinates just requiring that X does not depend on the choice of coordinates

$$(x')^a \frac{\partial}{\partial (x')^a} = x^b \frac{\partial}{\partial x^b} = x^b \left(\frac{\partial x'^a}{\partial x^b} \frac{\partial}{\partial x'^a} \right)$$

summed

X is the same in the x' and x coordinates

chain rule

$$\frac{\partial}{\partial x^b} = \frac{\partial x'^a}{\partial x^b} \frac{\partial}{\partial x'^a}$$

By equating the coefficients of $\frac{\partial}{\partial x'^a}$ we get

$$\boxed{x'^a = \left(\frac{\partial x'^a}{\partial x^b} \right) x^b}$$

If x'^a and x^a are related by a Lorentz transformation

we have $$\frac{\partial x'^a}{\partial x^b} = L^a{}_b \sim \text{constant}$$

which is the transformation rule we want.

Similarly an object with several upper indices (such as η^{ab}) that transforms as

$$T'^{a_1 \cdots a_n} = \left(\frac{\partial x'^{a_1}}{\partial x^{c_1}} \right) \cdots \left(\frac{\partial x'^{a_n}}{\partial x^{c_n}} \right) T^{c_1 \cdots c_n}$$

We call these objects $(n,0)$ tensors (and for $n=1$ they are just standard vectors).

We know that objects with one lower index transform with $(L^{-1})^a{}_b = \frac{\partial x^a}{\partial x'^b}$ (where now we see the x^a's as functions of the x'^b's).

Check $\left(\frac{\partial x^a}{\partial x'^b}\right) \cdot \left(\frac{\partial x'^b}{\partial x^c}\right) = \frac{\partial x^a}{\partial x^c} = \delta^a{}_c$ ✓

Then, in analogy with what is done above, it is convenient to introduce the dual vectors (or 1-forms) $x_a \, dx^a$. Again we can easily obtain the desired transformation law from here

$$x'_a \, dx'^a = x_b \, dx^b = x_b \frac{\partial x^b}{\partial x'^a} dx'^a \Rightarrow \boxed{x'_a = \frac{\partial x^b}{\partial x'^a} x_b}$$

A tensor with l lower indices transforms as

$$T'_{b_1 \dots b_l} = \left(\frac{\partial x^{d_1}}{\partial x^{b_1}}\right) \cdots \left(\frac{\partial x^{d_l}}{\partial x^{b_l}}\right) T_{d_1 \dots d_l}$$

and is called a $(0, l)$ tensor. Of course we can have mixed tensors which transform as

$$T'^{a_1 \dots a_k}{}_{b_1 \dots b_l} = \left(\frac{\partial x'^{a_1}}{\partial x^{c_1}}\right) \cdots \left(\frac{\partial x'^{a_k}}{\partial x^{c_k}}\right) \left(\frac{\partial x^{d_1}}{\partial x'^{b_1}}\right) \cdots \left(\frac{\partial x^{d_l}}{\partial x'^{b_l}}\right) T^{c_1 \dots c_k}{}_{d_1 \dots d_l}$$

There is a natural product between differentials and vectors (called inner product)

$$i_{\frac{\partial}{\partial x^a}}(dx^b) = dx^b\left(\frac{\partial}{\partial x^a}\right) \equiv \delta^a_b$$

In general tensors do not have fixed symmetries properties in their indices, i.e. $T_{a_1 a_2}$ is not related to $T_{a_2 a_1}$. Special classes of tensors are

- symmetric tensors (such as η_{ab}) which do not change when two indices are swapped
- antisymmetric tensor which change sign when two indices are swapped. An antisymmetric $(0,\ell)$ tensor is called an ℓ-differential form or ℓ-form for short. The electromagnetic field is encoded in a 2-form F_{ab}

$$F_{ab} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -cB_3 & cB_2 \\ -E_2 & cB_3 & 0 & -cB_1 \\ -E_3 & -cB_2 & cB_1 & 0 \end{pmatrix}$$

with $E_x = E_1$, $E_y = E_2$, $E_z = E_3$ and $B_x = B_1$, $B_y = B_2$, $B_z = B_3$.

Exercise: a) Show that the Maxwell equations in the vacuum can be written in the following relativistic form

$$\partial_{[a} F_{bc]} = 0 \quad \text{and} \quad \partial_a F^{ab} = 0$$

where the square parenthesis means that the indices are fully antisymmetrised i.e.

$$\partial_{[a} F_{bc]} = \frac{1}{6} \left(\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} - \right.$$
$$\left. \partial_a F_{cb} - \partial_b F_{ac} - \partial_c F_{ba} \right)$$

Actually the first and the second line on the rhs are identical since $F_{ab} = - F_{ba}$.

b) Find the transformation laws of $\underline{E}$ and $\underline{B}$ under a Lorentz boost along x^1 starting from the general transformation law of tensors.